January 2010 Algebra Prelim

Sarah Arpin

1

Show that if all subgroups of a group G are normal, then $[[G, G], G] = \{1\}$.
(Hint: By considering the action by conjugation of G on a cyclic subgroup $K \trianglelefteq G$, show that $[G, G] \leq C_G(K)$.)

Solution:
Let G act by conjugation on a cyclic subgroup $K \trianglelefteq G$. This action produces a homomorphism φ from G to $\mathrm{Aut}(K)$. Since K is cyclic, the automorphism group of K is abelian, so the image of G under this action, $\varphi(G)$, must also be abelian. By the first isomorphism theorem:

$$G/\ker\varphi \cong \varphi(G)$$

So the $G/\ker\varphi$ is abelian. Since $G/[G, G]$ is the smallest abelian quotient, this means $[G, G] \leq \ker\varphi$.
The kernel of this action is the set of all elements of G which commute with the elements of K, so $\ker\varphi = C_G(K)$. This shows that $[G, G] \leq C_G(K)$.
Since K was chosen as an arbitrary cyclic subgroup of G, this shows that $[G, G]$ is a subgroup of the center of every cyclic subgroup of G and we have:

$$[G, G] \leq \bigcap_{g \in G} C_G(\langle g \rangle) = Z(G)$$

So every commutator element of G commutes with every element of G. Consider an element $ghg^{-1}h^{-1} \in [[G, G], G]$, where $g \in [G, G]$ and take $h \in G$:

$$ghg^{-1}h^{-1} = gg^{-1}hh^{-1} = 1$$

Which proves the desired claim: $[[G, G], G] = \{1\}$.

$\square$

2

Which finite groups have exactly two automorphisms?

Solution:
If a finite group has exactly two automorphisms, then one of these must be the identity.
Note that $\mathbb{Z}_2$ has precisely two automorphisms: the identity, and the automorphism switching its two elements.
If the group has more than two elements, then the other, non-identity automorphism cannot be conjugation, because then we would get more than two automorphisms.
This means that the set of inner automorphisms must be $\{1\}$, so:

$$\text{Inn}(G) = G/Z(G) = \{1\}$$

This means that $Z(G) = G$, so G must be abelian. By the fundamental theorem of finitely generated abelian groups, this means that G can be expressed as a direct product.
The number of automorphisms of a cyclic abelian group G is $\varphi(|G|)$, where φ denotes the Euler ϕ-function.
If we require $\varphi(|G|) = 2$, then $|G| = 3, 4$, or 6.
The only abelian group of order 3 is $\mathbb{Z}_3$.
There are two abelian groups of order 4, but $|\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)| = 2(2-1)^2(2+1) = 6$, so only $\mathbb{Z}_4$ has the correct size automorphism group.
There is only one abelian group of order 6: $\mathbb{Z}_6$.
So the groups with precisely two automorphisms are the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$.

$\square$

3

Let R be a commutative ring with unity. Suppose for each $r \in R$, there exists an integer $n_r > 1$ such that $r^{n_r} = r$. Show that every prime ideal in R is maximal.

Solution:
Let R be a commutative ring with unity with the property described.
Let P be a prime ideal of R. Then R/P is an integral domain. To show that P is maximal, we must show that R/P is a field.
Consider $r + P \in R/P$. Then:

$$\begin{aligned}
r + P &= r^{n_r} + P \\
r(1 + P) &= r(r^{n_r - 1} + P) \\
1 + P &= r^{n_r - 1} + P \\
&= r(r^{n_r - 2} + P) \\
&= (r + P)(r^{n_r - 2} + P)
\end{aligned}$$

So the inverse of $r + P$ is $r^{n_r - 2} + P$. Since $r + P$ was chosen arbitrarily, this shows that every element of R/P has an inverse, so it is a field. R/P being a field then guarantees that P is a maximal ideal.

□

4

Determine the Jordan form of the $n \times n$ matrix over a field $\mathbb{F}$ whose entries are all 1's. (The answer depends on whether $\text{char}(\mathbb{F})$ divides n.)

Solution:
First, suppose $\text{char}(\mathbb{F}) \nmid n$.

5

Let p be a prime number. Let $\mathbb{F}$ be a field whose characteristic is not p which contains a primitive p-th root of unity. Suppose that $a, b \in \mathbb{F}$ are such that $\mathbb{F}[\sqrt[p]{a}] \neq \mathbb{F}[\sqrt[p]{b}]$. Prove that $\mathbb{F}[\sqrt[p]{a}, \sqrt[p]{b}] = \mathbb{F}[\sqrt[p]{a} + \sqrt[p]{b}]$.

Solution:
If either $\mathbb{F}[\sqrt[p]{a}]$ or $\mathbb{F}[\sqrt[p]{b}]$ are trivial, then $\mathbb{F}[\sqrt[p]{a}, \sqrt[p]{b}] = \mathbb{F}[\sqrt[p]{a} + \sqrt[p]{b}]$ is immediate.

6

Show that $x^5 - 4x + 2$ is not solvable by radicals over $\mathbb{Q}$.

Solution:
The polynomial $p(x) = x^5 - 4x + 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion using the prime $p = 2$. Consider $p'(x) = 5x^4 - 4 = (x^2\sqrt{5} - 2)(x^2\sqrt{5} + 2)$. The real zeros of this polynomial are $\pm\frac{\sqrt{2}}{\sqrt[4]{5}}$.