August 2007 Algebra Prelim

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1

Prove that in a group of order 12, any two elements of order 6 must commute.

Solution:
Let G be a group, $|G| = 12$.
If G is abelian, then every element commutes with every other element and we are done. Assume G is not abelian and take $x, y \in G$ with $|x| = |y| = 6$, and suppose $y \notin \langle x \rangle$.
$G = \langle x \rangle \langle y \rangle$, so this means that $|\langle x \rangle \cap \langle y \rangle| = 3$.
The subgroups of $\langle x \rangle$ and $\langle y \rangle$ of order 3 are $\{1, x^2, x^4\}$ and $\{1, y^2, y^4\}$, respectively, so if these are equivalent we either have $y^2 = x^2$ or $y^2 = x^4$.

2

Show that any group of order 105 has an element of order 35.

Solution:
Let G be a group, $|G| = 105$. First, consider the Sylow theorems. $105 = 5 \cdot 7$.
The number of Sylow 5-subgroups of G, n_5, must be 1, since

$$n_5 \equiv 1 \pmod 5, \text{ and } n_5 | 7$$

The number of Sylow 7-subgroups of G, n_7, must also be 1, since

$$n_7 \equiv 1 \pmod 7, \text{ and } n_7 | 5$$

Since there is only one Sylow 5-subgroup, it must be a normal subgroup of G: Every conjugate of a Sylow 5-subgroup of G must also be a Sylow 5-subgroup, so if $P_5 \in \mathrm{Syl}_5(G)$ is equal to all of its conjugates it is a normal subgroup of G.
The same holds for the unique Sylow 7-subgroup of G, so we have $P_7, P_5 \trianglelefteq G$.
This means that the product P_7P_5 is a subgroup of G. Also,

$$|P_7P_5| = \frac{|P_7||P_5|}{|P_7 \cap P_5|} = 35$$

Let $H = P_7P_5$. We will show that H has to be cyclic to establish that H has a generator, which will be our desired element of order 35.
Doing a Sylow analysis on H, we see that H is unique Sylow 5- and 7-subgroups. Since $|H| = 35$, H is the direct product of these subgroups, and this direct product is necessarily cylic, because it is isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_5$.
Thus, H has a generator of order 35, as desired.

$\square$

3

Let R be an integral domain in which every nonzero element factors into a product of finitely many irreducible elements up to a unit. For any $a, b \in R - \{0\}$, define the ideal:

$$I_{a,b} := \{x \in R : ax \in (b)\}$$

where (b) is the ideal of R generated by the element b. Then show that R is a UFD if and only if $I_{a,b}$ is principal for any $a, b \in R - \{0\}$.

Solution:
Then

4

Let R be an associative ring with $1 \neq 0$ and let $N \subseteq M$ be left R-modules. Suppose that N and M/N are Noetherian. Then show that M is Noetherian.

Solution:
If

5

6

Determine the splitting field of the polynomial $x^5 + 2x^4 + 5x^2 + x + 4$ and its Galois group.

Solution:
First, check to see if the polynomial has any linear factors. We do this by brute force.
$x = -2$ is a root.