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Math 2300: Calculus

Spring 2022

## 8.6: Representing Functions as Power Series

Lecture: Representing Functions with Power Series

## 8.6: Representing Functions with Power Series

Recall Geometric Series Information:

If  $|r| < 1$ , then the geometric series  $\sum_{n=0}^{\infty} ar^n$  converges. Recall that we have the formula:

$$\sum_{n=0}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

*first term of the series*  
*r common ratio*

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Recall Power Series Information:

The power series  $\sum_{n=0}^{\infty} a_n x^n$  was our first one we learned at:

Remember how we looked at it and recognized that it is a geometric series with  $r = x$ , so it converges precisely when  $x \in (-1, 1)$ .

When we assume  $x \in (-1, 1)$ , how can we use the formula  $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ ?

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } x \text{ in } (-1, 1)$$

*1 + x + x<sup>2</sup> + ...*

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Subtlety: Note that we can plug in any  $x \neq 1$  into the formula  $\frac{1}{1-x}$ , but it's only equal to the power series for that  $x$  value for  $x$ 's in  $(-1, 1)$ .

$$f(3) = \sum_{n=0}^{\infty} 3^n \quad \text{diverges}, \quad \text{but} \quad \frac{1}{1-3} = \frac{1}{-2}$$

*The fraction  $\frac{1}{1-x}$  doesn't know that it only works for  $x \in (-1, 1)$ .  
We need to know that the series diverges outside of this interval.*

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Back to the point  $\star \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{on } x \in (-1, 1)$

Directions that "look like"  $\frac{1}{1-x}$  can be written as power series.  
Let's just re-visit from "introduction to power series". The last two should be familiar:

Example 1:

Find a power series representation for the function  $\frac{2}{1+x}$ . For what values of  $x$  will this power series representation be valid?

$$\frac{2}{1+x} = 2 \cdot \frac{1}{1+x} = 2 \cdot \frac{1}{1-(-x)} = 2 \cdot \sum_{n=0}^{\infty} (-x)^n$$

geometric series, if  $r = -x$   
converges for  $|r| = |x| < 1$

$\rightarrow x \in (-1, 1)$

The power series representation of  $\frac{2}{1+x}$  is

$$\sum_{n=0}^{\infty} 2(-x)^n = \sum_{n=0}^{\infty} (2)(-1)^n x^n$$

and the function and power series are equal for  $x$ 's in  $(-1, 1)$ .

Example 2:

$$\frac{2}{1-2x} ?$$

Find a power series representation for the function  $\frac{2}{1-2x}$ . For what values of  $x$  will this power series representation be valid?

$$\text{Solution: } \frac{2}{1-2x} = 2 \cdot \frac{1}{1-2x} = 2 \cdot \frac{1}{3(1+\frac{2x}{3})} = 2 \cdot \frac{1}{3} \cdot \frac{1}{1+\frac{2x}{3}} = \frac{2}{3} \cdot \frac{1}{1-(\frac{-2x}{3})}$$

Power Series Rep.:  $\frac{2}{3} \sum_{n=0}^{\infty} (-\frac{2x}{3})^n = \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n (\frac{2}{3})^n x^n$

Solution:  $\frac{2}{3+2x} = 2 \cdot \frac{1}{3+2x} = 2 \cdot \frac{1}{3(1+\frac{2x}{3})} = 2 \cdot \frac{1}{3} \cdot \frac{1}{1+\frac{2x}{3}} = \frac{2}{3} \cdot \frac{1}{1-(\frac{-2x}{3})}$

Power Series Rep.:  $\frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n = \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{2}{3}\right)^n \cdot x^n$   
 $= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^{n+1} x^n$   
 $= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^n}{3^{n+1}}$

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Example 3:

$$\ln(5-x) ?$$

Find a power series representation for the function  $\ln(5-x)$ . What is the interval of convergence?

Solution:

Hint:

$$-\int \frac{1}{5-x} dx = \ln(5-x) + C.$$

$$\ln(5-x) + C = - \int \frac{1}{5-x} dx$$

$$= - \int \frac{1}{5} \cdot \frac{1}{1-\left(\frac{x}{5}\right)} dx$$

$$= -\frac{1}{5} \int \frac{1}{1-\left(\frac{x}{5}\right)} dx$$

$$= -\frac{1}{5} \int \left( \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n \right) dx$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} \left( \int \left(\frac{x}{5}\right)^n dx \right) = -\frac{1}{5} \left[ \int 1 dx + \int \left(\frac{x}{5}\right) dx + \int \left(\frac{x^2}{5^2}\right) dx + \int \left(\frac{x^3}{5^3}\right) dx + \dots \right]$$

$$\star \quad = -\frac{1}{5} \left[ x + \frac{x^2}{2 \cdot 5} + \frac{x^3}{3 \cdot 5^2} + \frac{x^4}{4 \cdot 5^3} + \dots + C \right]$$

$$\ln(5-x) = -\frac{1}{5} \left[ \underbrace{\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^{n-1}}} + C \right]$$

When does this series converge?

$$\text{Ratio test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)5^n} \cdot \frac{5^{n-1} \cdot n}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x \cdot n}{(n+1) \cdot 5} \right| = \left| \frac{x}{5} \right| < 1$$

• Check  $x=5$ :

$$\sum_{n=1}^{\infty} \frac{5^n}{n \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$$

$\frac{5}{n} < 1$  and  $\frac{5}{n} > -1$

$x < 5$  and  $x > -5$

diverges, harmonic series

• Check  $x=-5$ :

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 5}{n}$$

abs. harmonic series, so converges.

Interval of convergence:  $[-5, 5]$

$$\ln(5-x) = -\frac{1}{5} \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^{n-1}} + C$$

at  $x=4$ ,  $\ln(1)=0$ :

$$0 = -\frac{1}{5} \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 5^{n-1}} + C$$

$$\frac{1}{5} \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 5^{n-1}} = C \rightarrow \ln(5-x) = -\frac{1}{5} \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^{n-1}} + \frac{1}{5} \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 5^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{-x^n}{n \cdot 5^n} + \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 5^n}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{-x^n + 4^n}{n \cdot 5^n}} \quad \text{for } x \in [-5, 5]$$

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Lecture 8b: Section 8.6

Example 4:

Find a power series representation for the function  $\frac{1}{(1-x)^2}$ . For which values of  $x$  will this power series representation hold?

$$\frac{1}{(1-x)^2} = \frac{1}{1-2x+x^2}$$

Solution:

Hint:

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) \stackrel{I \circ C}{=} (-1, 1)$$

$$= \frac{d}{dx} (1 + x + x^2 + x^3 + x^4 + \dots)$$

$$= (1) + 2x + 3x^2 + 4x^3 + \dots$$

$$\frac{1}{(1-x)^2} = \boxed{\sum_{n=0}^{\infty} (n+1)x^n} \quad \leftarrow \text{Ratio test: } L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| x \frac{n+1}{n} \right| = |x| < 1$$

$$\boxed{x \in (-1, 1)} \quad I \circ C$$

$$\boxed{(1-x)}$$

$\xrightarrow{n \rightarrow \infty}$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{(n+2)}}{x^{(n+1)}} \right| = |x| < 1$$

$\boxed{x \in (-1, 1)}$  I.o.C.

Example 5:

Find a power series representation for the function  $\frac{2x^2}{1+x^3}$ . For what values of  $x$  will this power series representation hold?

Solution: 
$$\frac{2x^2}{1+x^3} = 2x^2 \cdot \frac{1}{1-(-x^3)} = 2x^2 \sum_{n=0}^{\infty} (-x^3)^n = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2}$$

Interval & conv.:  $(-1, 1)$

Proof:  
 $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , we know the series conv. abs.  
 $= 1$ , we do more investigation  
 $> 1$ , diverges

$$L = \lim_{n \rightarrow \infty} \left| \frac{2(-1)^{n+1} x^{3(n+1)+2}}{2(-1)^n x^{3n+2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{3n+5}}{x^{3n+2}} \right| = \lim_{n \rightarrow \infty} |x^3| = |x^3| < 1$$

$\downarrow \quad \downarrow$   
 $x^3 < 1 \quad \text{and} \quad x^3 > -1$   
 $(-1, 1) \leftarrow \text{abs. conv.}$

Check  $x = -1$ :  $\sum_{n=0}^{\infty} 2(-1)^n (-1)^{3n+2} = \sum_{n=0}^{\infty} 2 \cdot (-1)^{4n+2} = \sum_{n=0}^{\infty} 2 \text{ diverges by div. test.}$   
 $\lim_{n \rightarrow \infty} (2) = 2$ .

Check  $x = 1$ :  $\sum_{n=0}^{\infty} (2)(-1)^n 1^{3n+2} = \sum_{n=0}^{\infty} 2(-1)^n \text{ diverges by div. test.}$   
 $\lim_{n \rightarrow \infty} 2 \cdot (-1)^n \neq 0$   
(DNE.)