Friday: Power Series

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PowerSeries

Math 2300: Calculus **Spring 2022**

8.5: Power Series

Lecturer: Sarah Arpin

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$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

A **power series** is an "infinite degree polynomial". In other words, it's a series where we have a variable x (a variable besides the iteration variable). In general, it looks like:

$$\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

power series is a function of x:

→ we can evaluate it for different values of x

A particular example:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

We can talk about the convergence of a power series for particular values of x:

? Does this power series converge at $x=1$?

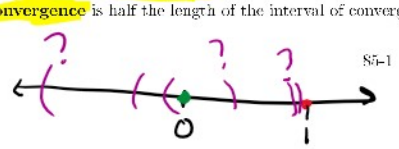
NO! p-series test w/ $p=1$, so diverges.

(at $x=1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$)

You should think of a power series as a **function**, where the input is some value that we provide for x, and the output is a series (a sum of numbers).

The **interval of convergence** is the interval of x-values for which a power series converges. The **radius of convergence** is half the length of the interval of convergence.

At $x=0$, it converges.



Introductory Example:

For what values of x does the power series $\sum_{n=0}^{\infty} x^n$ converge?

For each value of x, this power series will be a geometric series.

We know that a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges when $|r|<1$

So this power series converges when $|x|<1$.

$|x|<1$

$x<1$ AND $x>-1$

Plot this interval:

Interval of convergence: $(-1, 1)$

Radius of convergence: 1

Example 1:

What is the radius of convergence of the power series $\frac{1}{1}+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$?

interval and

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$0! = 1$

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$ ..

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

- Abs. conv. if $L<1$
- Div. if $L>1$
- Inconclusive if. $L=1$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!}\right| = L$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}\cdot n!}{(n+1)!\, x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$ for all values of x.

$L=0$ for all x in $(-\infty,\infty)$, so the power series converges (absolutely) for all x in $(-\infty,\infty)$. Radius of conv. $=\infty$

Example 2:

Find the interval and radius of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Ratio Test: $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^{n+1}}(4x-8)^{\cancel{n+1}}}{(n+1)}\cdot\frac{n}{\cancel{2^n}\cdot\cancel{(4x-8)^n}}\right|$

$= \lim_{n\to\infty}\left|\frac{n\boxed{(4x-8)\cdot 2}}{1n+1}\right| = |(4x-8)\cdot 2|$

same deg of n in num. and denom → ratio of the leading coeff.

$L = |(4x-8)\cdot 2|$

If $L<1$, we have abs. conv.
If $L=1$, we have uncertainty

$L<1$

$|8x-16|<1$

$8x-16<1$ AND $8x-16>-1$

$8x<17$ $\quad 8x>15$

$x<17/8$ $\quad x>15/8$

$15/8 \quad 2 \quad 17/8$

$L=1$

$|8x-16|=1$

$8x-16=-1$ or $8x-16=1$

$8x=15$ $\quad 8x=17$

$\boxed{x=15/8}$ $\quad \boxed{x=17/8} \to \ldots$

$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(\cancel{4}\cdot\frac{15}{\cancel{8}_{2}}-8\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{2^n\left(\frac{15}{2}-8\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(15-16)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ ← converges b/c alt. harmonic.

Example 3:

Find the interval and radius of convergence of the series:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{2^n}$$