

# Wednesday: Taylor Polynomials

Sunday, April 3, 2022 10:00 PM

PDF

TaylorPoly...

Properties of a polynomial approx. of a function:  
- Matching derivative values

Math 2300: Calculus

Spring 2022

8.7: Taylor Polynomials

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## 8.7: Taylor Polynomials (Continued)

Last class, we introduced Taylor polynomials as a way to approximate complicated functions. Let's formalize that now:

**Definition**

$T_n(x)$ , the nth degree Taylor polynomial for  $f(x)$  centered at  $x=a$ , is defined:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If  $a=0$ , we call  $T_n(x)$  a Maclaurin polynomial

**Noteworthy Property**

$T_n(x)$  evaluated at  $x=a$

What happens when you evaluate  $T_n(a)$ ?  $T'_n(a)$ ?  $T''_n(a)$ ?

$$T_n(a) = f(a) + \frac{f'(a)}{1!}(a-a) + \frac{f''(a)}{2!}(a-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(a-a)^n$$

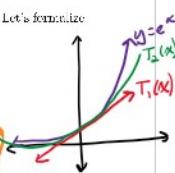
$$T_n(a) = f(a)$$

$$T'_n(x) = f'(a) \cdot 1 + \frac{2f''(a)}{2!}(x-a) + \dots + \frac{n f^{(n)}(a)}{n!}(x-a)^{n-1}$$

$$T'_n(a) = f'(a) + \frac{f''(a)}{1!}(a-a) + \dots + \frac{f^{(n)}(a)}{(n-1)!}(a-a)^{n-1}$$

$$T'_n(a) = f'(a)$$

\* The derivative values of  $f$  at  $x=a$  match the derivative values of  $T_n$  at  $x=a$ .



$$\frac{n}{n!} = \frac{1}{\underbrace{(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}_{(n-1)!}}$$

**Example 1**

- (c) Find  $T_6(x)$ , the 6th-degree Taylor polynomial for  $f(x) = \cos(x)$  centered at  $x = 0$ . Hint: Use a table to organize your work.

- (d) Use your polynomial to estimate  $\cos(5^\circ)$ .

- (e)  $\cos(x)$  is an even function ( $f(-x) = \cos(x)$ ). Is  $T_6(x)$ ?

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\cos(x) = 1$	1	$1 = f(0)$
1	$-\sin(x)$	0	0
2	$-\cos(x)$	-1	$-\frac{1}{2}$
3	$\sin(x)$	0	0
4	$\cos(x)$	1	$\frac{1}{24}$
5	$-\sin(x)$	0	0
6	$-\cos(x)$	-1	$-\frac{1}{720}$

*Maclaurin polynomial*  


$$T_6(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(6)}(0)}{6!}(x-0)^6 \quad w/ a=0.$$

$$\textcircled{a} \quad T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

$$\textcircled{b} \quad T_6(x) \approx \cos(x)$$

$$\cos(5^\circ) \approx T_6(5) = 1 - \frac{1}{2}5^2 + \frac{1}{24}(5)^4 - \frac{1}{720}5^6$$

$$\cos\left(\frac{\pi}{36}\right) \approx T_6\left(\frac{\pi}{36}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{36}\right)^2 + \frac{1}{24}\left(\frac{\pi}{36}\right)^4 - \frac{1}{720}\left(\frac{\pi}{36}\right)^6$$

$$\textcircled{c} \quad 0.99619\dots$$

**Example 2**

- (a) What is  $T_n(x)$ , the  $n$ th-degree Taylor polynomial for  $f(x) = \sin(x)$  centered at  $x = 0$ ?

- (b) Estimate  $\sin(2)$  using  $T_4(x)$ .

- (c) What could you do to improve your estimate in part (b)?

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\sin(x)$	0	0
1	$\star$	1	$\frac{1}{1} = 1$
2	$-\frac{1}{2}\cos(x)$	-1	$-\frac{1}{2!} = -\frac{1}{2}$
3	$\frac{1}{3}\sin(x)$	2	$\frac{2}{3!} = \frac{1}{3}$
4	$-\frac{1}{4}\cos(x)$	-3	$-\frac{3}{4!} = -\frac{1}{16}$
5	$\frac{1}{5}\sin(x)$	4	$\frac{4}{5!} = \frac{1}{60}$
$\vdots$	$\frac{(-1)^{n+1}}{n+1}\sin(x)$	$(-1)^n$	$\frac{(-1)^n}{n!} = \frac{(-1)^n}{n!}$

$$\boxed{T_n(x) = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} x^k}$$

$$\begin{aligned} & \text{← const.} \\ & \text{← coeff. of } x \\ & \text{← coeff. of } x^2 \\ & \vdots \\ & \text{← coeff. of } x^n \end{aligned}$$

$$\begin{aligned} & \text{← coeff. of } x \\ & \text{← coeff. of } x^2 \\ & \vdots \\ & \text{← coeff. of } x^n \end{aligned}$$

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Example:  $T_3(x) = 2 - 3x + \frac{1}{3}x^2 - \frac{1}{12}x^3$

What is  $f''(0)$ ?

No subtraction  $\Rightarrow a = 0$ .

$$\frac{f''(0)}{2!} (x-0)^2 \quad \left\{ \begin{array}{l} \text{appears in } T_3(x) \text{ above.} \\ \frac{1}{3}x^2 = \frac{f''(0)}{2!} (x-0)^2 \end{array} \right.$$

$$\frac{1}{3} = \frac{f''(0)}{2!}$$

$$\frac{2}{3} = f''(0)$$

**A Look Ahead**

If we construct an “infinite degree” Taylor polynomial, a lot of the time we can get *precisely* the function that we want. Now, polynomials are only allowed to be finite degree. So this is a stretch. But using something called a **power series**, which we will introduce soon, this is possible.