Monday: Integral Test Remainder Estimate

Sunday, April 3, 2022 8:37 PM

IntegralTe...

Math 2300: Calculus	**Spring 2022**

8.3 Integral Test Remainder Estimate

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Integral Test Remainder Estimate

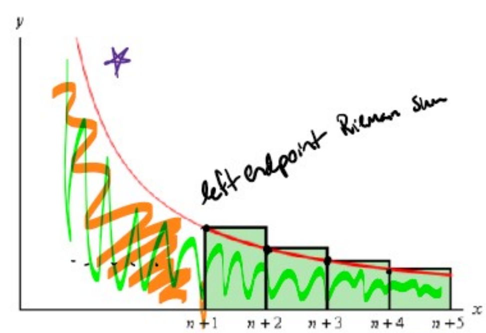
Recall the hypotheses required for the convergence of a series $\sum_{n=1}^{\infty} a_n$ to be determined by the integral test:

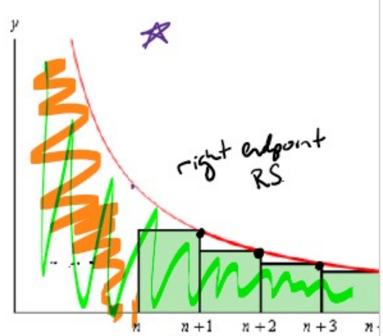
def. f(x) w/ f(n) = a_n

- f(x) decresing on $[1,\infty)$
- f(x) continuous on $[1,\infty)$
- f(x) positive on $[1,\infty)$

When these hypotheses are satisfied, we can conclude that $\sum_{n=1}^{\infty} a_n$ converges **if and only if** $\int_1^{\infty} f(x)dx$ converges.

For a convergent series, we can use partial sums to estimate the integral. What can the integral tell us about those partial sums?





$\sum_{k=1}^{n} a_k$] compare w/ $\int_{n+1}^{\infty} f(x)dx$

and $\int_{n}^{\infty} f(x)dx$

Theorem 83.1 (Integral Test Remainder Theorem) *If R_N estimates our error after an Nth partial sum:*

$$\int_{N+1}^{\infty} f(x)dx \le R_N \le \int_{N}^{\infty} f(x)dx$$

Warning: *This was all done assuming that the hypotheses of the integral test were satisfied and the series converges! If you want to use this remainder estimate, you need to know that the hypotheses of the integral test are satisfied, and prove that the series is convergent!*

Example

$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} \approx \ . \ . \ .$

1. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 5 terms. Use the integral test remainder estimate to estimate the error involved in this approximation.
2. How many terms are required to ensure the value of the sum is accurate to within 0.0005?

1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-test w/ $p = 3 > 1$.

Satisfies hyp of Integral Test.

1) continuous: $f(x) = \frac{1}{x^3}$ is discontinuous at $x=0$, so not in $[1,\infty)$.

2) decreasing: $\frac{1}{(n+1)^3} < \frac{1}{n^3}$ for all $n \ge 1$, b/c denom. is increasing and num = 1.

3) positive: $\frac{1}{n^3} > 0$ for all n: pos. denom & num.

→ By the integral test: $\underbrace{\int_6^{\infty} \frac{1}{x^3}dx}_{} \le |S - S_N| \le \underbrace{\int_5^{\infty} \frac{1}{x^3}dx}_{}$

$$= \lim_{L\to\infty} \int_6^L x^{-3}dx = \lim_{L\to\infty} \left(\frac{x^{-2}}{-2} \Big|_6^L \right)$$
$$= \lim_{L\to\infty} \left(\frac{1}{-2L^2} + \frac{1}{2\cdot 6^2} \right)$$
$$= \frac{1}{72}$$

$$= \lim_{L\to\infty} \int_5^L x^{-3}dx$$
$$= \lim_{L\to\infty} \left(\frac{1}{-2L^2} + \frac{1}{2\cdot 5^2} \right)$$
$$= \frac{1}{50}$$

$$\boxed{\frac{1}{72} \le |R_N| \le \frac{1}{50}}$$

2) $|S - S_N| = |R_N| \le 0.0005$

$$|S - S_N| \le \underbrace{\int_N^{\infty} f(x)dx}_{} \le 0.0005$$

$$= \lim_{L\to\infty} \int_N^L x^{-3}dx = \lim_{L\to\infty} \left(\frac{x^{-2}}{-2} \Big|_N^L \right) = \lim_{L\to\infty} \left(\frac{1}{-2L^2} + \frac{1}{2N^2} \right)$$

$$= \frac{1}{2N^2} \le 0.0005$$

$$\frac{1}{N^2} \le 0.001$$

$$\frac{1}{N^2} \le \frac{1}{1000}$$

$$1000 \le N^2$$

$$31.6 \le N \to \boxed{N = 32}$$

Example

$\sum_{n=1}^{\infty} n^{-1.001}$

Show that if we want to approximate the sum of the series ~~[illegible]~~ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

$\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

- Show the series satisfies Int. Test hyp.
- Converges by p-test w/ $p = 1.001 > 1$

$$\underbrace{\int_N^{\infty} x^{-1.001}dx}_{} \le .000000005$$

$$\underbrace{\int_N^{\infty} x^{-1.001}\,dx}_{} \le .000000005$$

$$= \lim_{L\to\infty}\int_N^L x^{-1.001}\,dx = \lim_{L\to\infty}\left(\frac{x^{-0.001}}{-0.001}\Bigg|_N^L\right)$$

$$= \lim_{L\to\infty}\left(\cancel{\frac{1}{-.001\cdot L^{.001}}} + \frac{1}{.001\cdot N^{.001}}\right)$$

$$\frac{1}{.001\cdot(N^{.001})} < .000000005$$