Tuesday: Ratio Test

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RatioTest

Math 2300: Calculus **Spring 2022**

8.4: Ratio Test

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Ratio Test

A test for **absolute convergence** or divergence.
Recall the definition of **absolute convergence:** Give an example of a series which is:

$\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ converges. (when this happens, $\sum a_n$ converges too)

1. Absolutely convergent: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is abs. conv. by p-test with $p=2>1$.
2. Conditionally convergent: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)
3. Divergent: $\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem 84.1 (The Ratio Test) *Consider the series* $\sum_{n=1}^{\infty} a_n$. *Let*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

1. *If* $L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is* ***absolutely convergent****.*
2. *If* $L > 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is* ***divergent****.*
3. *If* $L = 1$, *then the Ratio Test fails to draw a conclusion about the series* $\sum_{n=1}^{\infty} a_n$.

use another test!

How does the ratio test apply to a geometric series? Comparing a series to a geometric series is one way to prove the ratio test.

Don't use ratio test first:

$\sum_{n=1}^{\infty} n = 1+2+3+4+\ldots$ diverges by div. test: $\lim_{n\to\infty} n = \infty \neq 0$

Let's see what happens if we try to throw ratio test at this:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1$

inconclusive...

Example

$n! = n\cdot(n-1)\cdot(n-2)\cdot \cdots 3\cdot 2\cdot 1$

Determine the convergence of the series: $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

Factorial ⇝ Ratio Test?

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{100^{n+1}} \div \frac{n!}{100^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!\cdot \cancel{100^n}}{\cancel{100^{n+1}}\cdot n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\cdot\cancel{(n)}\cdot\cancel{(n-1)}\cdot\cancel{\cdots 3\cdot 2\cdot 1}}{100\cdot \cancel{n}\cdot\cancel{(n-1)}\cdot\cancel{(n-2)}\cdot\cancel{\cdots 3\cdot 2\cdot 1}}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{100} = \infty > 1$$

By the Ratio Test, $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges.

Example

Determine the convergence of the series: $\sum_{n=1}^{\infty} \overbrace{\frac{5^{n-1}}{(-2)^{n+1}n}}^{a_n}$

- Alt. series test?
- Limit comparison test?

Try ratio test: $L = \lim_{n\to\infty}\left|\frac{5^{n+1-1}}{(-2)^{n+1+1}(n+1)}\cdot\frac{(-2)^{n+1}n}{5^{n-1}}\right|$

$$= \lim_{n\to\infty}\left|\frac{5^{\cancel{n}}\cancel{(-2)^{n+1}}\cdot n}{(-2)^{\cancel{n+1}}(n+1)\cdot\cancel{5^{n-1}}}\right| = \lim_{n\to\infty}\left|\frac{5n}{-2(n+1)}\right|$$

$$= \lim_{n\to\infty}\frac{5n}{2n+2} = \frac{5}{2} > 1$$

Ratio Test Tells us the series diverges.

Example

Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \boxed{\frac{(-5)^n}{(n-1)!}} \leftarrow a_n$$

Factorials → Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{n+1}}{(n+1-1)!}\cdot\frac{(n-1)!}{(-5)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-5)^{n+1}\cdot(n-1)!}{n!\cdot(-5)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-5)^{\cancel{n+1}}\,\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{n\,\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}\cdot\cancel{(-5)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-5}{n}\right| = \lim_{n\to\infty}\frac{5}{n} = \underline{0<1}$$

absolutely converges by ratio test

Factorials:

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$

$n! = n\cdot(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1$

Example

For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$$

start w/ $k=2$:

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!} \quad \text{Ratio test: } L = \lim_{n\to\infty}\left|\frac{((n+1)!)^2}{(2(n+1))!}\cdot\frac{(2n)!}{(n!)^2}\right|$$

$(n+1)! = (n+1)\cdot\cancel{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}$

$n! = \cancel{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!\cdot(n+1)!\cdot(2n)\cdot(2n-1)\cdot(2n-2)\cdots 3\cdot 2\cdot 1}{(2n+2)!\cdot n!\cdot n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(n+1)\cdot\cancel{(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}}{(2n+2)(2n+1)\cdot\cancel{(2n)\cdot(2n-1)\cdots 3\cdot 2\cdot 1}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$$

converges by Ratio Test