

# Alternating Series

Wednesday, March 16, 2022 11:11 AM



8.4AtSeries

Math 2300: Calculus

Spring 2022

## 8.4: Alternating Series

Lecturer: Sarah Arpin

### Alternating Sequence

Remember that sequences are lists, and series are what you get when you add it up.

When we first introduced sequences, we encountered a few different types of alternating sequences:

$a_n = (-1)^n a_n = \cos(n\pi)$ , etc.  $\cos(\pi) = -1$   
 $a_n = \sin(n\pi + \frac{\pi}{2})$   $\cos(2\pi) = 1$   
 $\cos(3\pi) = -1$

We had an alternating sequence convergence test:  $a_n = \sin(n\pi + \frac{\pi}{2})$

not series

If  $\lim_{n \rightarrow \infty} |a_n| \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

We can use this to conclude that  $a_n = \frac{(-1)^n}{n}$  converges. Now let's see if there are similar applications for series that sum up the terms of an alternating sequence:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \checkmark$$

### Alternating Series

An alternating series will be something of the form:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}, \text{ or } \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \text{ or even } \sum_{n=1}^{\infty} \frac{(-1)^n (n+7)}{5 \cdot 2^n} \text{ or } -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

The identifying factor is that the terms of the series will alternate  $+, -, +, -, \dots$

To test convergence of such a series, we have a new tool, called the **alternating series test**. This test has two hypotheses, and if these hypotheses are satisfied we can conclude convergence:

Let  $a_n$  be an alternating sequence. Then:

Need to show

- if  $|a_n|$  are decreasing and
- $\lim_{n \rightarrow \infty} |a_n| = 0$ ,

then the series  $\sum_{n=1}^{\infty} a_n$  converges. } conclusion

\* This test cannot be used to prove an alternating series diverges.  
\* If you suspect an alt. series diverges, try div. test first.

**First Big Conclusion:**

Remember the harmonic series?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Last week, we concluded this series diverged by using the integral comparison test. (We won't go through that again today.)

There is something called the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Let's see if this one converges or diverges...

The sequence  $\frac{(-1)^{n+1}}{n}$  is alternating and  $|\frac{(-1)^{n+1}}{n}| = \frac{1}{n}$ .

The terms  $\frac{1}{n}$  are decreasing:  $\frac{1}{n+1} < \frac{1}{n}$  and

$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . So by the alternating series test,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges.

*Warning* - we are not making any kind of conclusion about  $\sum_{n=1}^{\infty} \frac{1}{n}$ . We are using the  $\frac{1}{n}$  terms to make a conclusion about the alt. series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ .

**Techniques to Show a Sequence is Decreasing:**

If  $a_n$  is a sequence of positive numbers, we can show that it is decreasing by showing one of the following:

1.  $a_{n+1} < a_n$
2.  $\frac{a_{n+1}}{a_n} < 1$
3.  $a_n - a_{n-1} > 0$
4. If we define a continuous differentiable function on  $[1, \infty)$  such that  $f'(x) = a_n$ , then  $a_n$  is decreasing iff  $f'(x) < 0$  on  $[1, \infty)$ .

*use these tools to show  $a_n$  is dec. in the alt. series test.*

**More Examples:**

1. Determine if the series

converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

*is an alternating series.*

$$|\frac{(-1)^n}{\sqrt{n}}| = \frac{1}{\sqrt{n}} \text{ is decreasing: } \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ . By the alt. series test,  $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$  converges.

2. Determine if the series

converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n}$$

*alternating series.  $|\frac{(-1)^{n+1}}{5^n}| = \frac{1}{5^n}$*

$$\sum_{n=1}^{\infty} 1 \text{ diverges by divergence test: } \lim_{n \rightarrow \infty} 1 = 1 \neq 0$$

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - \dots$$

$$= \lim_{N \rightarrow \infty} \sum_{n=1}^N (-1)^n$$

*does not exist*

$S_1 = -1$   
 $S_2 = 0$   $(-1+1)$   
 $S_3 = -1+1-1 = -1$   
 $S_4 = 0$

What about  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ?  
 Diverges by p-test w/  
 $p = \frac{1}{2} \leq 1$ .

2. Determine if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

converges or diverges.

alternating series.  $\left| \frac{(-1)^n n^2}{n^2+5} \right| = \frac{n^2}{n^2+5}$

look at  $f(x) = \frac{x^2}{x^2+5}$ :  $f'(x) = \frac{(x^2+5)(2x) - x^2(2x)}{(x^2+5)^2}$

$f'(x) = \frac{2x^3+10x-2x^3}{(x^2+5)^2} = \frac{10x}{(x^2+5)^2} < 0$  for  $x < 0$ ...

$\lim_{n \rightarrow \infty} \frac{(-1)^n n^2}{n^2+5} = \lim_{n \rightarrow \infty} \frac{(-1)^n}{1 + \frac{5}{n^2}} = \lim_{n \rightarrow \infty} (-1)^n$  does not exist... so  $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$  diverges by the divergence test.

3. Determine if the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

converges or diverges.

is an alternating series

Consider  $\left| \frac{(-1)^n \ln(n)}{n} \right| = \frac{\ln(n)}{n}$

• Is  $\frac{\ln(n)}{n}$  decreasing?  $\frac{\ln(n+1)}{n+1} < \frac{\ln(n)}{n}$ ?

look at  $f(x) = \frac{\ln(x)}{x}$   $f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2}$

$f'(x) = \frac{1 - \ln(x)}{x^2}$

$f'(x) < 0$  when  $1 - \ln(x) < 0$

$1 < \ln(x)$   
 $e < x$

So  $\frac{\ln(n)}{n}$  seq. is decreasing for all  $n \geq 3$

(This is ac, b/c  $\sum_{n=2}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ )

•  $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = \frac{\infty}{\infty}$ , so l'Hospital's applies

$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0$  ✓

So by the alternating series test,  $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$  converges.