

P Test Integral Test

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Math 2300: Calculus

Spring 2022

8.3: p Series and Integral Test

Friday March 11

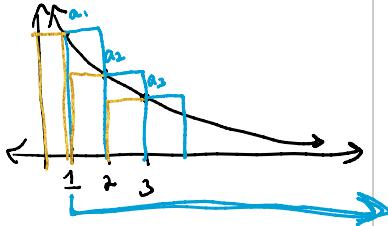
The Integral Test

$$\sum_{i=1}^{\infty} a_i, \quad \text{let } f(i) = a_i.$$

Big idea: Compare an infinite series to an integral to determine convergence/divergence.
If we define $f(x)$ such that $f(n) = a_n$, we can use the integral test when f is:

- continuous on $[1, \infty)$
- positive on $[1, \infty)$
- decreasing on $[1, \infty)$

When these hypotheses are satisfied, we can conclude:



- If $\int_1^\infty f(x)dx$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

- If $\int_1^\infty f(x)dx$ diverges, then $\sum_{i=1}^{\infty} a_i$ diverges.

Example:

Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{e^n} = \frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \dots$$

Our intuition looking at the numerator and denominator tells us we should probably expect convergence: e^n grows much faster than n . Let's use the integral test to prove it.

1. First, define $f(x) = \frac{x}{e^x}$.
2. $f(x)$ is continuous on $[1, \infty)$, since it is a function given by a polynomial over an exponential function, and so its denominator is nonzero.
3. $f(x)$ is positive for $x \geq 1$, since both numerator and denominator are ≥ 0 . *sign chart?*
4. To see that $f(x)$ is decreasing, let's take the derivative and show that it is always negative:

$$f'(x) < 0 \quad f'(x) = \frac{e^x - e^x \cdot x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}}$$

To see when this is negative, set up an inequality and consider it on the domain $[1, \infty)$:

$$0 > \frac{e^x - e^x \cdot x}{e^{2x}} \\ 0 > e^x(1-x)$$

Since e^x is always positive, the inequality above is equivalent to $0 > 1-x$ or $x > 1$. So $f'(x) = 0$ only when $x = 1$; $f'(x)$ is negative for any $x > 1$, which means f is decreasing on $[1, \infty)$.

5. Now, we just determine whether or not the integral converges to make a decision about the series.

$$\int_1^{\infty} f(x) dx = \lim_{T \rightarrow \infty} \int_1^T \frac{x}{e^x} dx$$

Use integration by parts

Let $u = x$, $dv = e^{-x} dx$, so that

$du = dx$ and $v = -e^{-x}$:

$$\begin{aligned} &= \lim_{T \rightarrow \infty} \left(-xe^{-x} \Big|_1^T - \int_1^T e^{-x} dx \right) \\ &= \lim_{T \rightarrow \infty} \left(-Te^{-T} - e^{-1} + e^{-T} \Big|_1^T \right) \\ &= \lim_{T \rightarrow \infty} \left(-Te^{-T} - e^{-1} + -e^{-T} + e^{-1} \right) \\ &= \frac{2}{e} + \lim_{T \rightarrow \infty} \frac{-(T+1)}{e^T} \\ &= \frac{2}{e} + \underset{\infty}{\text{indeterminate form, so use l'Hopital's rule}} \\ &= \frac{2}{e} + \lim_{T \rightarrow \infty} \frac{-1}{e^T} \\ &= \frac{2}{e} \end{aligned}$$

which is finite, so the integral converges. Thus, by the integral test, we can conclude that the series converges.

If it was $f(x) = \frac{e^x}{x}$:

- e^x is continuous everywhere,
 - x is continuous everywhere,
 - and the denominator x is only zero at $x=0$.
- $\Rightarrow f(x) = \frac{e^x}{x}$ is cont. on $(0, \infty)$

Example

Determine whether the series converges or diverges:

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1} = \frac{1}{2} + \frac{4}{9} + \frac{9}{28} + \dots$$

$$f(x) = \frac{x^2}{x^3+1}$$

1) Show $f(x)$ is positive on $[1, \infty)$: x^2 is never negative for $x \geq 1$, the fraction $\frac{x^2}{x^3+1}$ is $\frac{+}{+}$ so $f(x)$ is pos. on $[1, \infty)$

2) Show $f(x)$ is continuous on $[1, \infty)$: denom = 0 when $x = -1$, and the numerator and denominator are both polynomials, so continuous.
 $\Rightarrow f(x)$ is cont. on $[1, \infty)$

3) Show $f(x)$ is decreasing on $[1, \infty)$:

$f(x)$ is decreasing where $f'(x) < 0$:

$$f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{-x^4 + 2x}{(x^3+1)^2} < 0$$

always pos.

$$\begin{array}{c} -x^4 + 2x < 0 \\ x(6x^3 + 2) < 0 \end{array} \quad \begin{array}{c} -1 \leftarrow \\ 0 \\ 1 \leftarrow \color{blue}{\sqrt[3]{2}} \\ 2 \leftarrow \\ 3 \leftarrow \end{array}$$

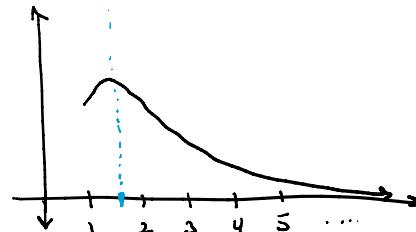
$f(x)$ is decreasing on $[\sqrt[3]{2}, \infty)$

$f(1) = \frac{1}{2}$, which is finite, and $f(x)$ is decreasing $[\sqrt[3]{2}, \infty)$

4) Integral: $\int_1^{\infty} \frac{x^2}{x^3+1} dx = \lim_{T \rightarrow \infty} \int_1^T \frac{x^2}{x^3+1} dx$

$u = x^3+1$
 $du = 3x^2 dx$
 $\frac{1}{3} du = x^2 dx$

$$\begin{aligned} &= \lim_{T \rightarrow \infty} \frac{1}{3} \int_1^T \frac{du}{u} \\ &= \lim_{T \rightarrow \infty} \frac{1}{3} \left(\ln|x^3+1| \right) \Big|_1^T = \lim_{T \rightarrow \infty} \frac{1}{3} (\ln|T^3+1| - \ln(2)) \end{aligned}$$



By the integral test, the series diverges.

Example

Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

***p* Series Test**

Let's use this to make conclusions about series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

→ Converges if $p > 1$
diverges if $p \leq 1$

for $p > 0$.

1. First define $f(x) = \frac{1}{x^p}$
2. Continuous: denom = 0 when $x=0$, and num is a const., so cont. denom. is a polynomial, so
3. Positive: 1 is pos, $x^p > 0$ for $x > 0$, so $f(x)$ is pos. continuous, so $f(x)$ is cont. on $[1, \infty)$
4. Decreasing: $f'(x) = -px^{p-1}$ is negative for $x > 0$, so $f(x)$ is dec. on $[1, \infty)$
5. Integral:

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{T \rightarrow \infty} \int_1^T x^{-p} dx \quad \begin{matrix} \text{Case 1, } p \neq 1 \\ \downarrow \text{Case 2: } p = 1 \end{matrix} \quad \lim_{T \rightarrow \infty} (\ln|x| \Big|_1^T) = \lim_{T \rightarrow \infty} (\ln(T) - \ln(1)) = \infty$$

$$\lim_{T \rightarrow \infty} \frac{x^{-p+1}}{-p+1} \Big|_1^T = \lim_{T \rightarrow \infty} \left(\frac{T^{-p+1} - 1}{-p+1} \right) = \begin{cases} \frac{1}{-p+1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

Example

Determine whether the series is convergent or divergent:

$$\frac{1}{1} + \frac{1}{4\sqrt[3]{2}} + \frac{1}{9\sqrt[3]{3}} + \frac{1}{16\sqrt[3]{4}} + \frac{1}{25\sqrt[3]{5}} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^{2+\frac{1}{3}}} = \sum_{k=1}^{\infty} \frac{1}{k^{2+\frac{1}{3}}}$$

$p = 2 + \frac{1}{3} > 1$, so the series converges by the p -test.