Series

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8.2Series

Math 2300: Calculus	**Spring 2022**

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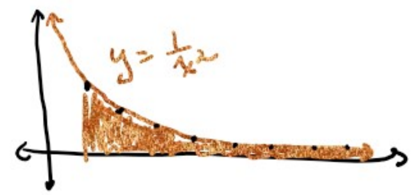
Lecturer: Sarah Arpin

Series

Indefinite Integrals :



Recall Notation

If you want to write out a sum:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

you may want to save space. To do this, you can use $\sum$ notation:

$$\overset{\text{stop}\rightarrow}{}\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \quad (\text{start} \rightarrow i=1)$$

What about if the thing you're adding up is more complicated? Just figure out how to use that "counter" i to your advantage:

$$\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} = \sum_{i=2}^{5} \frac{i}{i+1}.$$

We can also use this to add up the terms of a sequence:

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n}$$

Recall that our sequences can be infinite. How do we add them up in this case?
Take a sequence $a_1, a_2, \ldots$. Adding up all of the (infinitely many) terms gives us an (infinite) **series**:

1, 2, 3, 4, ... sequence

1 + 2 + 3 + 4 + ... sum

list of values

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots$$

one value determined by the sum of values in a sequence.

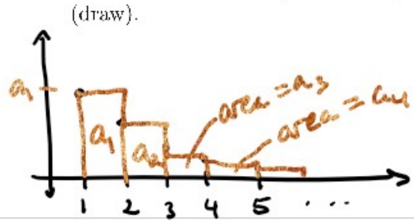
Notice the difference between a sequence and a series. In a sequence, we just list all the terms. In a series, we add them all up.
How do we do this infinitely many times? **Limits.** The value of an infinite series is **defined** by its **partial sums**:

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \boxed{\sum_{i=1}^{N} a_i}$$

N^{th} partial sum – stop at term N

- If $\sum\limits_{i=1}^{\infty} a_i$, we say the series **converges.** is finite
- If $\sum\limits_{i=1}^{\infty} a_i$ is infinite or DNE, we say the series **diverges.**
- We can think of this geometrically by imagining an infinite series as the integral of a step function (draw).

$f(n) = a_n$



Geometric Series

$a=1$, $r=2$	$1+2+4+8+\cdots$

$\lim_{N\to\infty} \frac{1-2^N}{1-2}$

$a=\frac{1}{3}$, $r=-2$	$\frac{1}{3}-\frac{2}{3}+\frac{4}{3}-\frac{8}{3}+\cdots$

$$\underbrace{a+ar+ar^2+ar^3+\cdots}=\boxed{\sum_{n=1}^{\infty} ar^{n-1}}$$

converges if $|r|<1$
diverges if $|r|\geq 1$

To investigate convergence, subtract the partial sums s_N and rs_N and solve for s_N:

$$S_N=\sum_{n=1}^{N} ar^{n-1} = \underbrace{a}+\cancel{ar}+\cancel{ar^2}+\cancel{\cdots}+\cancel{ar^{N-2}}+\cancel{ar^{N-1}}$$

$$r\cdot S_N = r\sum_{n=1}^{N} ar^{n-1} = \cancel{ar}+\cancel{ar^2}+\cancel{ar^3}+\cdots+\cancel{ar^{N-1}}+\underbrace{ar^N}$$

$$\underbrace{S_N - rS_N} = a - ar^N$$

$$S_N(1-r) = a-ar^N$$

$$\boxed{S_N=\frac{a-ar^N}{1-r}}$$ partial sum of a geo series

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty}\sum_{n=1}^{N} ar^{N-1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty}\frac{a-ar^N}{1-r}$$

When does this limit converge?

If $|r|<1$, then $\lim_{N\to\infty} r^N$ will converge, and so will our series

☆Take $r=\frac{1}{2}$ and see what happens

If the geo series converges, the value will be:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty}\frac{a-ar^N}{1-r}$$ ($ar^N \to 0$)

$$=\boxed{\frac{a}{1-r}}$$

Example

Is the series $\sum_{n=1}^{\infty} 5^{1-n}2^{2n}$ convergent or divergent?

Want in the form: $\sum_{n=1}^{\infty} ar^{n-1}$

$$5^{1-n} = (5^{-1})^{n-1} \text{ or } \left(\frac{1}{5}\right)^{n-1}$$

$$2^{2n}=(2^2)^n = 4^n = 4^{(n-1)+1} = 4^1\cdot 4^{n-1}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1}\cdot 4\cdot (4)^{n-1} = \sum_{n=1}^{\infty}(4)\cdot\left(\frac{4}{5}\right)^{n-1}$$

is a geometric series w/ $a=4$, $r=\frac{4}{5}$. Since $r=\frac{4}{5}$, $|r|<1$ and the series converges

Example

Is the series $\sum_{n=1}^{\infty}\frac{12}{3^n}$ convergent or divergent? If it is convergent, find the sum.

Want: $\sum_{n=1}^{\infty} a\cdot r^{n-1}$

$$\frac{12}{3^n} = 12\cdot\left(\frac{1}{3}\right)^n = \underbrace{12\cdot\frac{1}{3}}\cdot\left(\frac{1}{3}\right)^{n-1}$$

$$=4\cdot\left(\frac{1}{3}\right)^{n-1}$$

$$\to = \sum_{n=1}^{\infty} 4\cdot\left(\frac{1}{3}\right)^{n-1}$$

geometric series $a=4$ $r=\frac{1}{3}$

Since $r=\frac{1}{3}$, $|r|<1$ and the series converges

The sum is $\frac{a}{1-r}=\frac{4}{1-\frac{1}{3}}$

$$=\frac{4}{2/3}$$

$$={}^{2}\cancel{4}\cdot\frac{3}{\cancel{2}} = 6$$

Telescoping Series

Consider the series:

$$\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)}= \lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)$$

$$= \lim_{N\to\infty}\left[\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\cdots+\left(\cancel{\frac{1}{N-2}}-\cancel{\frac{1}{N-1}}\right)+\left(\cancel{\frac{1}{N-1}}-\cancel{\frac{1}{N}}\right)+\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)\right]$$

$$= \lim_{N\to\infty}\left(1-\frac{1}{N+1}\right) = \boxed{1} \qquad \boxed{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1}$$

Example

Express the following series as a telescoping sum and determine if it converges or diverges:

$$\underbrace{\sum_{k=1}^{\infty}\frac{2}{k^2-4k+3}}$$

$$\frac{2}{(k+3)(k+1)} = \left(\frac{A}{k+3}+\frac{B}{k+1}\right)$$

$$2 = A(k+1) + B(k+3)$$

$k=-1$: $2 = B(-1+3)$

$2 = 2B \longrightarrow B=1$

$k=-3$: $2 = A(-3+1)$

$2 = A(-2) \longrightarrow A=-1$

$$\sum_{k=1}^{\infty}\frac{2}{k^2+4k+3} = \boxed{\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)}$$

$$= \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)$$

$$= \lim_{N\to\infty}\left(\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\cdots+\left(\cancel{\frac{1}{N-1}}-\cancel{\frac{1}{N+1}}\right)+\left(\cancel{\frac{1}{N}}-\frac{1}{N+2}\right)+\left(\cancel{\frac{1}{N+1}}-\frac{1}{N+3}\right)\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\cancel{\frac{1}{N+2}}^{\to 0}-\cancel{\frac{1}{N+3}}^{\to 0}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$$

$\boxed{\text{converges to } 5/6}$

Series with Variables

Example

Find the value(s) of x for which the following series converges:

$$\underbrace{\sum_{i=1}^{\infty} \frac{(x+3)^i}{2^i}}_{\sum_{i=1}^{\infty} a \cdot r^{i-1}}$$

$(x+3)^i = (x+3)(x+3)^{i-1}$

$2^i = 2 \cdot 2^{i-1}$

$$= \sum_{i=1}^{\infty} \left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}\right)^{i-1}$$

is geometric w/ $a = \frac{x+3}{2}$

$r = \frac{x+3}{2}$

Geometric series converge for $|r| < 1$

$\left|\frac{x+3}{2}\right| < 1$

$-1 < \frac{x+3}{2} < 1$

$-2 < x+3 < 2$

$-5 < x < -1$

The series converges for $\boxed{-5 < x < -1}$.

Expressing a decimal as a series

Example

Express the repeating decimal $0.\overline{14}$ as a series.

$.\underbrace{14}_{100\text{ths}}14\,14\ldots = \frac{14}{100} + \frac{14}{10{,}000} + \frac{14}{10^6} + \ldots$

$= \frac{14}{10^2} + \frac{14}{10^4} + \frac{14}{10^6} + \ldots$

$= \boxed{\sum_{n=1}^{\infty} \frac{14}{10^{2n}}}$ geo!

$14 \cdot \underbrace{\frac{1}{10^{2n}}}$

$\frac{1}{10^{2n}} = \left(\frac{1}{100}\right)^n = \frac{1}{100} \cdot \left(\frac{1}{100}\right)^{n-1}$

$\sum_{n=1}^{\infty} \frac{14}{100}\left(\frac{1}{100}\right)^{n-1}$

geo w/ $a = \frac{14}{100}$, $r = \frac{1}{100}$

converges b/c $|r| < 1$

$\frac{a}{1-r} = \frac{14/100}{1 - 1/100} = \boxed{\frac{14}{99}}$