Sequences

Tuesday, March 1, 2022 11:14 AM

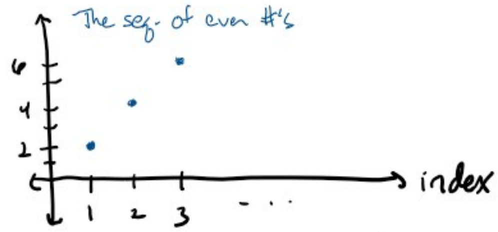
8.1Sequen...

Math 2300: Calculus **Spring 2022**

8.1 Sequences

Lecturer: Sarah Arpin

WebAssign due tonight



Sequences

1, 2, 3, ...
2, 4, $\boxed{6}$, 8, ...

Think of a **sequence** as a list of numbers:

$$a_1, a_2, a_3, ..., a_k, ...$$

It is a function whose domain is positive integers:

$$f(1) = a_1,\ f(2) = a_2,\ ...,\ f(k) = a_k, ...$$

EX, in the sequence of even #'s $f(1) = 2$, $f(2) = 4$, ...

In Calc 1, we used limits to describe the behavior of function. Limits also apply to sequences.

Example:

Consider the sequence:

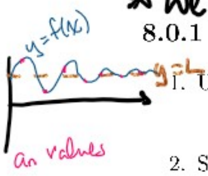
$$\overset{a_1}{1}, \overset{a_2}{\frac{1}{4}}, \overset{a_3}{\frac{1}{9}}, ..., \overset{a_k}{\frac{1}{k^2}}, ...$$

So $f(x) = \frac{1}{x^2}$ is the function that gives us this sequence, when we take the domain to be positive integers. What is the end behavior of this sequence?

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

★ We care about limits of sequences

8.0.1 Tools To Remember



1. Using functions to determine the limit of a sequence: If $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$, then...
   $\lim_{n\to\infty} a_n = L$

2. Squeeze Law (Sandwich Theorem): If $f(x) \le h(x) \le g(x)$ for all x and
   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = C,$$
   then... $\lim_{x\to\infty} h(x) = C$.
   Also for sequences: $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = C$ then, $\lim_{n\to\infty} b_n = C$.

Terms alternate neg./positive: $-1, 1, -1, 1, -1, \ldots$ ✓
$-1, 1, 1, -1, 1, 1, \ldots$ ✗

3. Showing an alternating sequence converges:
   If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$
   seq. of pos. values; alternating seq.

4. Leading Coefficient Test from limits of rational functions:
   If $f(x)$ and $g(x)$ are polynomials with the same degree, then $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$ = the ratio of their leading coefficients. **To prove** this:

Example sequence $a_n = \frac{n^3+2n+1}{2n^3-1}$ related function: $\boxed{f(x) = \frac{1x^3+2x+1}{2x^3-1}}$

$\lim_{x\to\infty} f(x) = \frac{1}{2}$

$\lim_{n\to\infty} \frac{n^3+2n+1}{2n^3-1}$ multiply num and denom. by $\frac{1}{n^3}$ ← largest power

$= \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^3+2n+1)}{\frac{1}{n^3}(2n^3-1)}$

$= \lim_{n\to\infty} \frac{1+\cancel{\frac{2}{n^2}}^{\to 0}+\cancel{\frac{1}{n^3}}^{\to 0}}{2-\cancel{\frac{1}{n^3}}^{\to 0}}$ $\lim_{n\to\infty}\frac{1}{n} = 0$

$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

$\frac{f(x)}{g(x)}$ and
... $\deg f(x) < \deg g(x) \to \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
... $\deg f(x) = \deg g(x) \to \lim_{x\to\infty} \frac{f(x)}{g(x)} =$ ratio of leading coeff's
... $\deg f(x) > \deg g(x) \to \lim_{x\to\infty} \frac{f(x)}{g(x)} = +$ or $-\infty$

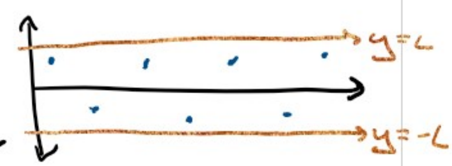
Three ways to show a sequence is decreasing:

consecutive terms

1. If $a_n > a_{n+1}$ for all n, then the sequence is decreasing.

2. If $a_n - a_{n+1} > 0$ for all n, then the sequence is decreasing.

3. If $\frac{a_{n+1}}{a_n} < 1$ for all n, then the sequence is decreasing.



Some Vocab

A sequence is **bounded** if there exists some number L such that $|a_n| \leq L$ for all L.
A sequence is **monotonic** if it is either strictly increasing for all x or strictly decreasing for all x.
Bounded monotonic sequences must converge. (Think about why?)

A **recursively** defined sequence is one where the definition of one term depends on the previous ones. You may be familiar with "the" Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ... where $f_n = f_{n-1} + f_{n-2}$.

3, 4, 7, 11, ...

Examples

Example 1:

Write an expression for a_n for the sequence that begins

$$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \ldots$$

$a_1 = \frac{2}{3 = 3^1}$

$a_2 = \frac{4}{9 = 3^2}$

$a_3 = \frac{6}{27 = 3^3}$

$a_4 = \frac{8}{81 = 3^4}$

$$\boxed{a_n = \frac{2n}{3^n}}$$

Strategy: Find a way of expressing the numerator in terms of what term it is (n), and do the same for the denominator.

Example 2:

Suppose $a_n = \underbrace{\frac{(-1)^n \ln(n)}{n}}_{\text{alternating!}}$. Find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} \left|\frac{(-1)^n \ln(n)}{n}\right| = \lim_{n\to\infty} \frac{\ln(n)}{n}$$

related function: $f(x) = \frac{\ln(x)}{x}$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x} \xrightarrow{} \frac{\infty}{\infty} = \text{apply l'H:} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Since a_n is alternating and $\lim_{n\to\infty}|a_n| = 0$, we conclude $\lim_{n\to\infty} a_n = 0$

Example 3:

Suppose $a_n = \frac{\sqrt{3n^2+4}}{n-1}$. Find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+4}}{n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^2}} \cdot \sqrt{3n^2+4}}{1 - \frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{3 + \cancel{\frac{4}{n^2}}^{\,0}}}{1 - \cancel{\frac{1}{n}}^{\,0}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{3}}{1} = \sqrt{3}$$

Example 4:

Suppose $a_n = \left(1+\frac{1}{n}\right)^n$. Find $\lim_{n\to\infty} a_n$.
This one has a trick to it, but it should be familiar...Let's use $f(x) = \left(1+\frac{1}{x}\right)^x$, so that we can use L'H if necessary.
Let's call this limit L, and then we can make use of log's:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = L$$

$$\ln(L) = \lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right)$$

$$\ln(L) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \quad \infty \cdot 0$$

$$\ln(L) = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x}) \to 0}{\left(\frac{1}{x}\right) \to 0}$$

$\frac{0}{0}$ indet. form. use L'Hopital's

$$\ln(L) = \lim_{x\to\infty} \frac{\frac{1}{(1+\frac{1}{x})}\cdot\left(\cancel{-\frac{1}{x^2}}\right)}{\left(\cancel{-\frac{1}{x^2}}\right)}$$

$$\ln(L) = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}, \quad \frac{1}{x}\to 0$$

$$\ln(L) = 1$$

raise both sides as powers of e to solve for L

$$e^{\ln(L)} = e^1$$

$$L = e$$

Example 5:

Show $a_n = \frac{3^{n+2}}{5^n}$ is decreasing.

$$a_n = \frac{3^{n+2}}{5^n} \qquad a_{n+1} = \frac{3^{(n+1)+2}}{5^{n+1}} = \frac{3^{n+3}}{5^{n+1}}$$

1) $\frac{a_{n+1}}{a_n} \overset{?}{<} 1$:

$$\frac{a_{n+1}}{a_n} = \left(\frac{3^{n+3}}{5^{n+1}}\right)\left(\frac{5^n}{3^{n+2}}\right)$$

$$\frac{a_{n+1}}{a_n} = \frac{3}{5} < 1$$, so the seq. is dec.

2) $a_n - a_{n+1} \overset{?}{>} 0$

$$a_n - a_{n+1} = \frac{3^{n+2}}{5^n}\cdot\frac{5}{5} - \frac{3^{n+3}}{5^{n+1}} = \frac{5\cdot 3^{n+2}}{5^{n+1}} - \frac{3^{n+3}}{5^{n+1}} = \frac{5\cdot 3^{n+2} - 3\cdot 3^{n+2}}{5^{n+1}} = \frac{2\cdot 3^{n+2}}{5^{n+1}} > 0$$

so the seq. dec.

Example 6:

Show $a_n = \frac{n}{n+1}$ is increasing.

Plan:
To show increasing, we want to show $a_{n+1} > a_n$.
This follows if we can show $\frac{a_{n+1}}{a_n} > 1$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{3}{4}$

$$a_n = \frac{n}{n+1} \qquad a_{n+1} = \frac{n+1}{n+2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \div \frac{n}{n+1}$$

$$= \frac{(n+1)^2}{(n+2)n} = \frac{n^2+2n+1}{n^2+\ldots} > 1 \text{ b/c}$$

$$= \frac{(n+1)^2}{(n+2)n} = \frac{n^2+2n+1}{n^2+2n} > 1$$ b/c numerator is greater than denom.

so a_n is increasing.