Monday 2/21

Sunday, February 20, 2022 11:39 PM

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Math 2300: Calculus	**Spring 2022**

Calculus with Parametric Equations

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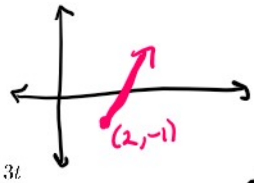
Warm-Up

A ray is parametrized by

$x(0) = 2$
$y(0) = -1$
$m = \frac{5}{3}$

$$x(t) = 2 + 3t$$
$$y(t) = -1 + 5t$$



$\frac{x-2}{3} = t$
$\hookrightarrow y = -1 + 5\left(\frac{x-2}{3}\right)$

where $t \geq 0$. *Note that we are imposing this restriction on t!
If we hadn't said anything, we would consider the full line $t \in (-\infty, \infty)$.

(a) Does $(5, 4)$ lie on the ray?

(b) Does $(2, 1)$ lie on the ray?

(c) Does $(-1, -6)$ lie on the ray?

(d) When does the line hit the y-axis?

(e) What is the speed of the motion along the line?

(f) What is the slope of the line?

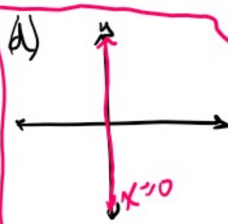
x, y

(a) $4 = -1 + 5t \rightarrow 5 = 5t \rightarrow t = 1$ $\quad x(1) = 2 + 3 = 5$
yes, (5,4) occurs at $t = 1$ for this ray

(b) $2 = 2 + 3t \rightarrow t = 0 \rightarrow y(0) = -1$ NO.

(c) $-1 = 2 + 3t \rightarrow -3 = 3t \rightarrow t = -1$ No, $t = -1 < 0$, so not on the ray

(d)



y-intercept is at $x = 0$.

$0 = 2 + 3t$
$-2 = 3t$
$-2/3 = t < 0$
$\rightarrow$ do not cross y-axis.

(e) $\text{speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$

$x'(t) = 3$
$y'(t) = 5$

$\text{speed} = \sqrt{3^2 + 5^2}$
$= \boxed{\sqrt{34}}$

(f) $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \boxed{\frac{5}{3}}$

The Calculus of Parametric Equations, Part II

Recall:

- We are thinking of parametric equations as describing the path taken by a snail. Think: t = time
- We have talked about describing this path in two ways: parametric and cartesian equations.
- We have talked about the various speeds associated to the path: $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dy}{dx}$. instantaneous speed = $\sqrt{(x'(t))^2+(y'(t))^2}$

Something else we might be interested in:

- Calculating a **second derivative** $\frac{d^2y}{dx^2}$
- Calculating how far our snail has travelled: This is called **arc length.**
- Calculating the **average speed** of a snail along a path.

Second Derivative

If $y=f(x)$

second derivative formula for parametric equations

$$f''(x) = \frac{d^2y}{dx^2} = \overbrace{\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \boxed{\frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}}$$

$\frac{d^2y}{dx^2} = y''$ ← concavity of the graph $y=f(x)$

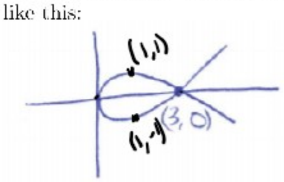
The first equality is chain rule: We temporarily think of t as a function of x. The second equality is just simplifying.

This equation still tells us about the concavity of the path. This is useful, because some of the paths cross like this:

$$\boxed{\frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}} \ldots$$

$>0 \Rightarrow y=f(x)$ is concave up

$<0 \Rightarrow y=f(x)$ is concave down



so it's useful to talk about the concavity at a *time* instead of at an x-value.
Let's put it to use!

Example

Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.

$$x(t) = 1 - t^2$$
$$y(t) = t^7 + t^5$$

$*\ x'(t) = -2t$ denom

$y'(t) = 7t^6 + 5t^4$

$$\frac{y'(t)}{x'(t)} = \frac{7t^6+5t^4}{-2t}$$

$$\boxed{\frac{y'(t)}{x'(t)} = \frac{7}{-2}t^5 - \frac{5}{2}t^3}$$

$$\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{35}{-2}t^4 - \frac{15}{2}t^2$$

numerator

$$\frac{d^2y}{dx^2} = \frac{\frac{35}{-2}t^4 - \frac{15}{2}t^2}{-2t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(\frac{35}{-2}t^3 - \frac{15}{2}t\right)$$

concave up

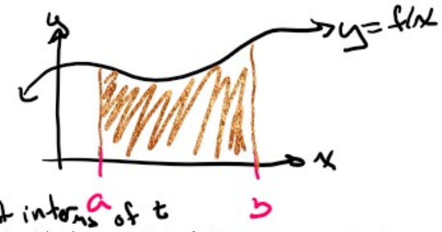
factor out GCD

$$\frac{d^2y}{dx^2} = \frac{35}{4}t^3 + \frac{15}{4}t > 0$$

$$\underbrace{\frac{5}{4}t}_{\substack{>0 \text{ if } t>0 \\ <0 \text{ if } t<0}}\underbrace{(7t^2+3)}_{\substack{\text{always be } \geq 3, \\ \text{so certainly } >0}} > 0 \Rightarrow \boxed{\begin{array}{l}\text{concave up } t>0 \\ \text{concave down } t<0\end{array}}$$

$\rightarrow y = f(x)$

Area Enclosed by Parametric Curves



The area under the curve $y = F(x)$ from a to b is given

$$A = \int_{\alpha=t}^{\beta=t} y(t)\, x'(t)\, dt$$

y-vals

$$A = \int_a^b F(x)\,dx.$$

want in terms of t

If the curve is traced out by the parametric equations $x(t)$ and $y(t)$, for $\alpha < t < \beta$, then we can calculate the area:

depend on t

$$A = \int_a^b y\,dx = \int_\alpha^\beta y(t)x'(t)\,dt.$$

y-vals in terms of t

$x(t)$

$$\frac{dx}{dt} = x'(t)$$

$$\boxed{dx = x'(t)\,dt}$$

Example

Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$, and the y-axis.

$x=0$

$$0 = t^2 - 2t$$
$$0 = t(t-2)$$
$$t=0,\ t=2$$

curve crosses the y-axis @ $t=0$, $t=2$.

$$x'(t) = 2t - 2$$

$$\sqrt{t} = t^{1/2}$$

$$A = \int_0^2 \sqrt{t}\,(2t-2)\,dt$$

$$= \int_0^2 \left(2t^{3/2} - 2t^{1/2}\right) dt$$

$$= \frac{2t^{5/2}}{5/2} - \frac{2t^{3/2}}{3/2}\Bigg|_0^2$$

$$= \frac{4}{5}2^{5/2} - \frac{4}{3}2^{3/2}$$

$$= \frac{4}{5}\sqrt{32} - \frac{4}{3}\sqrt{8}$$

$$\sqrt{32} = \sqrt{16}\,\sqrt{2} \quad (\sqrt{16} = 4)$$

$$\sqrt{8} = \sqrt{4}\cdot\sqrt{2} \quad (\sqrt{4} = 2)$$

$$= \frac{16}{5}\sqrt{2} - \frac{8}{3}\sqrt{2}$$

$$= \frac{48}{15}\sqrt{2} - \frac{40}{15}\sqrt{2} = \boxed{\frac{8}{15}\sqrt{2}}$$

Average Speed

This one should be familiar. We know that the function that gives us the speed is:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

We recall that the average value of a function on $[a, b]$ is given by:

$$\frac{1}{b-a}\int_a^b f(x)\,dx,$$

so **average speed** is just a special case of this. The average speed on a parametric function is:

$$\frac{1}{b-a}\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

a = t-value
b = t-value

Example

Find the average speed of the snail whose path is given

$$x(t) = 3\sin(t)$$

$$y(t) = 3\cos(t)$$

from $\underset{a}{t = 0}$ to $\underset{b}{t = \pi}$.

$$\text{Avg speed} = \frac{1}{b-a}\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$\to x'(t) = 3\cos(t)$$

$$y'(t) = -3\sin(t)$$

$$\text{Avg. speed} = \frac{1}{\pi - 0}\int_0^{\pi} \sqrt{\underbrace{(3\cos(t))^2}_{} + \underbrace{(-3\sin(t))^2}_{}}\,dt$$

$$9\cos^2(t) + 9\sin^2(t)$$

$$= \frac{1}{\pi}\cdot 3\int_0^{\pi}\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{=1}}\,dt$$

$$= \frac{3}{\pi}\int_0^{\pi} 1\,dt = \frac{3}{\cancel{\pi}}\cdot\cancel{\pi} = \boxed{3}$$

Arc Length

Arc length is the integral of speed: The length of the arc between $t = a$ and $t = b$ on the path whose parametric equations is given by $x(t)$ and $y(t)$ is given:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example

Determine the length of the parametric curve given by the following parametric equations:

$$x(t) = 3\sin(t)$$

$$y(t) = 3\cos(t)$$
$$0 \leq t < 2\pi$$