Friday: Calculus with Parametric Equations

Saturday, February 12, 2022 8:40 PM

CalcOfPara...

Math 2300: Calculus **Spring 2022**

Section 6.4

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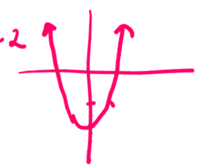
Warm-Up

Consider the parametric equations:

$$x(t) = 3t - 1$$
$$y(t) = t^2 - 2$$

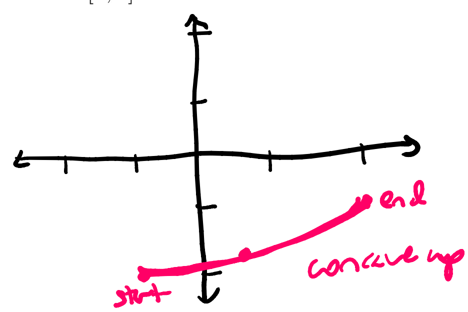
$y = x^2 - 2$



(a) By plugging in values for t, sketch the shape of the curve for $t \in [0, 1]$.
Solution:

t	$x(t) = 3t-1$	$y(t) = t^2-2$
0	-1	-2
1	2	-1
$1/2$	$1/2$	$-7/4$ or -1.75

$x'(t) = 3$ $\quad$ $y'(t) = 2t$



(b) Eliminate the parameter to find a Cartesian equation of the curve.
Solution:

(c) Does this shape make sense with what you sketched?

$(\underbrace{x(t)}, \underbrace{y(t)})$ $x'(t)$ $y'(t)$ $\frac{dy}{dx}$

The Calculus of Parametric Equations

1. $x'(t) = \frac{dx}{dt}$ = the instantaneous velocity in the x direction — rate of change in x w/ resp. to the parameter t

2. $y'(t) = \frac{dy}{dt}$ = the instantaneous velocity in the y direction

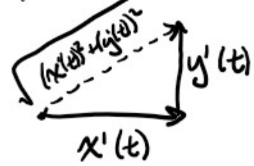
3. ★ $\frac{dy}{dx}$ = the rate of change in y with respect to x; the slope of the tangent line

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}}$$

We can describe the velocity of the snail in the x-direction. We can describe the velocity of the snail in the y-direction. What about the overall speed of the snail?

The **instantaneous speed** of the snail along the curve as a function of t is given:

4. $$\boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$



This follows from the Pythagorean theorem! Think about why.

Examples

Consider the parametric equations:

$$x(t) = t^3 - 4$$
$$y(t) = 4t - t^2$$

1. Find an equation of the tangent to the curve at the point corresponding to $t = 1$.

$y - y_1 = m(x - x_1)$
1) → (x_1, y_1) is a pt. on the line
2) → m is the slope of line

① point on the line:
$x(1) = 1 - 4 = -3$
$y(1) = 4 - 1 = 3$
→ $(-3, 3)$

② slope:
$m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ at $t = 1$

$\left.\frac{4-2t}{3t^2}\right|_{t=1} = \frac{2}{3}$

$m = 2/3$

$\boxed{y - 3 = \frac{2}{3}(x + 3)}$

2. Find the point(s) on the curve where the tangent is horizontal.

$\frac{dy}{dx} = \frac{4-2t}{3t^2}$

horizontal tangent → $m = 0$

$\frac{4-2t}{3t^2} = 0$ $\quad 4 - 2t = 0, \ t = 2$

$x(2) = 8 - 4 = 4$
$y(2) = 8 - 4 = 4$
$(4, 4)$

3. Graph on a graphing calculator for an appropriate interval of t-values to check.

Example ✯ $\sin^2\theta + \cos^2\theta = 1$

Suppose the position (x, y) of a particle at time t is given by the parametric equation

$$\begin{cases} x(t) = 4\sin(t), \\ y(t) = 2 - \cos(t), \end{cases} \quad \begin{aligned} &\rightarrow \frac{x}{4} = \sin(t) \\ &2 - y = \cos(t) \end{aligned} \qquad \left(\frac{x}{4}\right)^2 + (2-y)^2 = 1$$

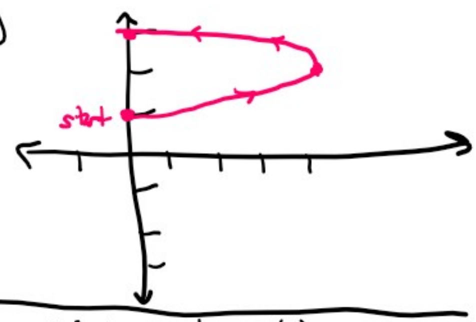
for $0 \leq t \leq \pi$.

(a) Eliminate the parameter t to find a cartesian equation for the path traced by the particle. $\left(\frac{x}{4}\right)^2 + (2-y)^2 = 1$

(b) Draw a graph to depict the motion of the particle for $0 \leq t \leq \pi$. On your graph, mark the start and end points and the direction of motion of the particle.

(c) What is the instantaneous speed of the particle at $t = \pi/2$?

(b)



	t	$x(t) = 4\sin(t)$	$y(t) = 2-\cos(t)$
Start	0	0	1
	$\pi/2$	4	2
end	π	0	3

$\sqrt{(x'(t))^2 + (y'(t))^2} \rightarrow x'(t) = 4\cos(t)$, $y'(t) = \sin(t)$

$x'(\pi/2) = 0 \quad y'(\pi/2) = 1 \rightsquigarrow$ inst. speed $= \sqrt{0^2 + 1^2} = 1$

units: (units for x and y)/(units for time)

Class question:

$y = \sin(t)$
$x = \cos^2(t)$

$x + y^2 = 1$

Example

Sketch $x(t) = t\sin(t)$, $y(t) = t\cos(t)$ for $0 \le t \le 3\pi$.