

Tuesday: Center of Mass

Saturday, February 12, 2022 8:39 PM



CenterOf...

Math 2300: Calculus

Spring 2022

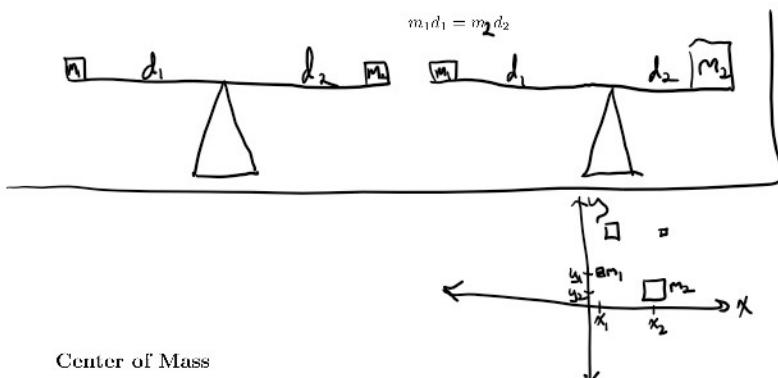
Section 6.6: Tuesday February 15

Lecturer: Sarah Arpin

Moments and Center of Mass

Work is translational force, moments are rotational force. <https://youtu.be/22VGQM1jCn8>

Law of the Lever



Center of Mass

The **center of mass** (or centroid, or center of moments) is the balance point of an object (a flat 2D object, or a 1D object).

For a system of n particles with masses m_1, m_2, \dots, m_n located at the points $(x_1, y_1), \dots, (x_n, y_n)$ in the plane, the **center of mass of the system** is located at (\bar{x}, \bar{y}) , where:

Center of mass
for a system of
finitely many points:

$$\bar{x} = \frac{x_1 m_1 + \dots + x_n m_n}{m_1 + \dots + m_n}$$
$$\bar{y} = \frac{y_1 m_1 + \dots + y_n m_n}{m_1 + \dots + m_n}$$

} weighted average

- Where would you place your finger to balance it?
- What if the mass of one side was heavier than the other? Then where would you place your finger?

Examples:

1. Find the center of mass of the system of the following point masses:

- A mass of 5 at (1, 4)
- A mass of 2 at (3, -2)
- A mass of 10 at (-1, -4)

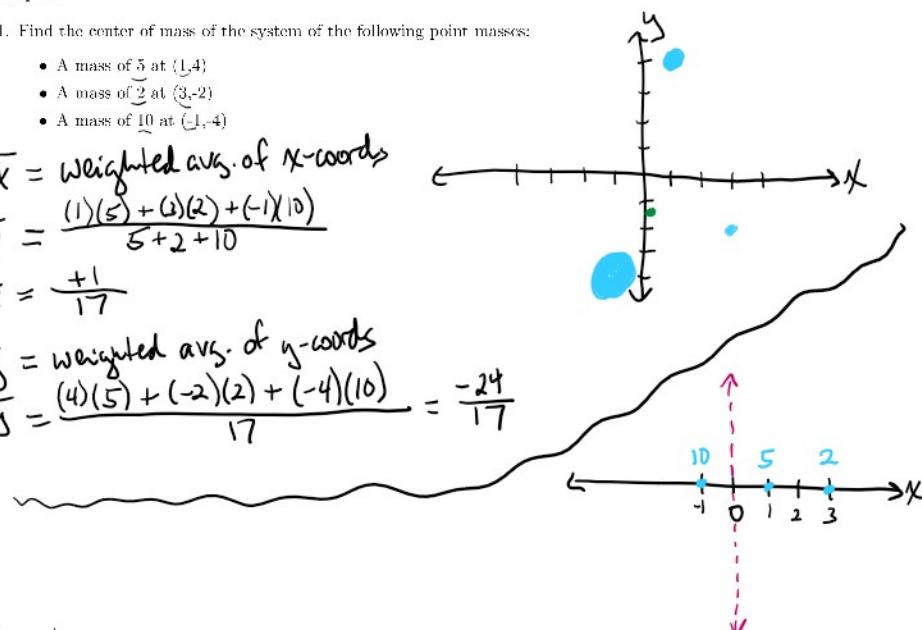
$$\bar{x} = \text{Weighted avg. of } x\text{-coords}$$

$$\bar{x} = \frac{(1)(5) + (3)(2) + (-1)(10)}{5+2+10}$$

$$\bar{x} = \frac{+1}{17}$$

$$\bar{y} = \text{Weighted avg. of } y\text{-coords}$$

$$\bar{y} = \frac{(4)(5) + (-2)(2) + (-4)(10)}{17} = \frac{-24}{17}$$



Moments

The **moment of the system about the y-axis**, denoted M_y , is the tendency of the system to rotate about the y-axis.

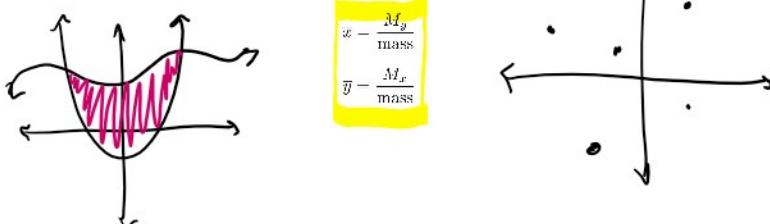
The **moment of the system about the x-axis**, denoted M_x , is the tendency of the system to rotate about the x-axis. Their equations are:

$$\bar{x}\text{-numerical } M_y = x_1 m_1 + \dots + x_n m_n = \sum_{i=1}^n x_i m_i$$

$$\bar{y}\text{-numerical } M_x = y_1 m_1 + \dots + y_n m_n = \sum_{i=1}^n y_i m_i$$

x is the distance from the y-axis \rightarrow tendency to rotate around the y-axis

Notice that M_x and M_y appear in our equations for the COM:



Putting it all together

What if we have a thin region bounded by curves and we want to find the center of mass? Say $f(x)$, $g(x)$ bounds a region between $x = a$ and $x = b$, with $f(x) > g(x)$ on $[a, b]$. (Sketch). Then: take a small slice. What is the center of mass of this slice?

$$\bar{x} = x$$

$$\bar{y} = \frac{f(x) + g(x)}{2}$$

$$\bar{x} = x$$

$$\bar{y} = \frac{f(x) + g(x)}{2}$$

$$\tilde{y} = \frac{f(x) + g(x)}{2}$$

What is the mass of this slice?

$$\text{mass} = \underbrace{\text{Area}}_{\text{mass}} \cdot (\text{Mass density of region}) = [f(x) - g(x)] \cdot dx \cdot \rho$$

Putting this together with out knowledge that infinite sums are just integrals, we get:

$$\bar{x} = \frac{M_x}{\text{total mass}} = \frac{\int_a^b \tilde{x} \cdot \text{(mass)}}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\rho \int_a^b x(f(x) - g(x))dx}{\rho \int_a^b (f(x) - g(x))dx} = \boxed{\frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}}$$

$$\bar{x} = \frac{\sum m_i x_i}{m_1 + m_2 + \dots + m_n}$$

$$\bar{x} = \frac{M_x}{\text{mass}} = \frac{\int_a^b \tilde{x} \cdot \text{(mass)}}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\rho \int_a^b \frac{f(x) + g(x)}{2} (f(x) - g(x))dx}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\frac{1}{2} \int_a^b (f(x) + g(x))(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

We can simplify this a bit as:

$$\bar{x} = \frac{\int_a^b (f^2(x) - g^2(x))dx}{2 \int_a^b (f(x) - g(x))dx}$$

This gives us a way to compute centers of mass of bounded regions between curves. Let's use it! But before that, what's with ρ cancelling out? Since we are assuming the density is uniform across the object, it basically doesn't matter what it is! The center of mass will be the same if it's really dense vs. not dense, because the COM is just a balancing point.

Example 1:

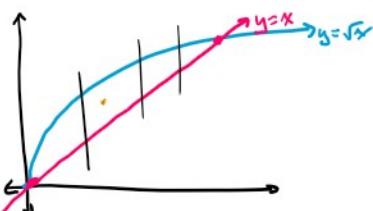
Find the centroid of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx} = \frac{\int_a^b (f(x)^2 - g(x)^2) dx}{2 \int_a^b (f(x) - g(x)) dx}$$

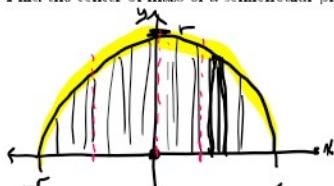
$$\textcircled{1} \int_0^1 x(\sqrt{x} - x) dx = \int_0^1 (x^{3/2} - x^2) dx = \left[\frac{x^{5/2}}{5/2} - \frac{x^3}{3} \right]_0^1 = \frac{1.2}{5} - \frac{1}{3}$$

$$\textcircled{2} \int_0^1 (\sqrt{x} - x) dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\bar{x} = \frac{\frac{1.2}{5}}{\frac{1}{6}} = \frac{6}{5} = \frac{2}{3}$$



Example 2:

Find the center of mass of a semicircular plate of radius r .

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx}$$

$$\bar{x} = \frac{\int_{-r}^r x(r^2 - x^2) dx}{\int_{-r}^r (r^2 - x^2) dx}$$

$\bar{x} = 0$ by symmetry
or b/c integral of odd
func. across sym. interval is 0.

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = \pm \sqrt{r^2 - x^2}$$

Top semi-circle: $y = \sqrt{r^2 - x^2}$

$$\bar{y} = \frac{\int_a^b (f(x)^2 - g(x)^2) dx}{2 \int_a^b (f(x) - g(x)) dx}$$

$$\bar{y} = \frac{0}{2 \int_{-r}^r \sqrt{r^2 - x^2} dx} = \frac{0}{2 \cdot \frac{1}{2} \pi r^2} = 0$$

$$\bar{y} = \frac{4r^3/3}{2(4\pi r^3/3)} = \frac{4r}{3\pi}$$

$$f(x) = \sqrt{r^2 - x^2}$$

$$g(x) = 0$$

$$\textcircled{1} \int_{-r}^r (r^2 - x^2) dx = r^2 x - \frac{x^3}{3} \Big|_{-r}^r = \left(r^2(r) - \frac{r^3}{3} \right) - \left(r^2(-r) - \frac{(-r)^3}{3} \right) = r^3 - r^3/3 + r^3 - r^3/3 = 2r^3 - 2r^3/3 = \frac{4r^3}{3} - \frac{2r^3}{3} = 4r^3/3$$

$$\textcircled{2} \int_{-r}^r \sqrt{r^2 - x^2} dx = \int_{-r}^r r \cos \theta \cdot r \cos \theta d\theta = \int_{-r}^r r^2 \cos^2 \theta d\theta = r^2 \int_{-r}^r \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= \frac{r^2}{2} \int_{-r}^r (1 + \cos(2\theta)) d\theta = \frac{r^2}{2} \left(\theta + \frac{\sin(2\theta)}{2} \right) \Big|_{-r}^r = \frac{r^2}{2} \left(\arcsin\left(\frac{r}{r}\right) + \frac{2\sin(2\theta)\cos\theta}{2} \Big|_{-r}^r \right)$$

$$= \frac{r^2}{2} \left(\arcsin\left(\frac{r}{r}\right) + \frac{2\sin(2\theta)\cos\theta}{2} \Big|_{-r}^r \right) = \frac{r^2}{2} \left(\arcsin\left(\frac{r}{r}\right) + \frac{2\cdot 0}{2} \right) = \frac{r^2}{2} \left(\arcsin(1) + \frac{2\cdot 0}{2} \right) = \frac{r^2}{2} \left(\arcsin(1) - \arcsin(-1) \right) = \frac{r^2}{2} \left(\arcsin(1) - (-\arcsin(1)) \right) = \frac{r^2}{2} (2\arcsin(1)) = \frac{r^2 \pi}{2}$$

Section 6.6

$$\text{COM: } (0, \frac{4r}{3\pi})$$

Example 3:

Find the moments M_x and M_y of the triangle connecting the points $(0,0)$, $(2,4)$, and $(2,0)$.

$$\begin{aligned} M_x &= \int_{-r}^r y \cdot \text{base} \cdot \text{height} dx = \int_{-r}^r y \cdot 2 \cdot (4 - y) dx \\ &= 2 \int_{-r}^r y(4 - y) dx = 2 \int_{-r}^r (4y - y^2) dx = 2 \left[4x - \frac{x^3}{3} \right]_{-r}^r = 2 \left[4r - \frac{r^3}{3} \right] - 2 \left[-4r - \frac{(-r)^3}{3} \right] = 2 \left[4r - \frac{r^3}{3} \right] + 2 \left[4r + \frac{r^3}{3} \right] = 2 \cdot 8r = 16r \end{aligned}$$

$$\begin{aligned} \frac{\pi}{2} \sum_{n=1}^{\infty} &= \frac{c^2}{2} \operatorname{arctanh}^{(1)} \\ &= \frac{c^2}{2} \left(\frac{\pi}{2} - \frac{\pi}{2} \right) = \frac{c^2 \pi}{2} \end{aligned}$$