05.05

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Math 1300: Calculus I **Fall 2020**

Lecture: Section 5.5: The Substitution Rule

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$x\int_a^b 1\,dx$
$= x(b-a)$

Today's Goal: Learn an integration technique called "u-substitution"

Logistics: We start this on a Tuesday (12/1), after an activity, and finish on Wednesday.

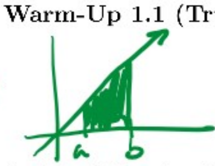
Warm-Up 1.1 (True or False:) *If $f(x)$ is continuous on $[a,b]$, then:*

$\int_a^b 2f(x)dx = 2\int_a^b f(x)dx$

works for constants but not variables.

$\int c f(x)dx = c\int f(x)dx$

$\int_a^b x\,dx$:



$$\int_a^b x f(x)dx = x\int_a^b f(x)dx.$$

False.

1.1 Chain Rule

With all of our antiderivative practice, we still have difficulty un-doing chain rule, especially more complicated versions. For Example:

$$\boxed{\frac{d}{dx}\sin(x^2-2x) =} \cos(x^2-2x)\cdot(2x-2)$$

Which means we can actually integrate:

$$\int (2x-2)\cos(x^2-2x)dx = \sin(x^2-2x) + C$$

But if we had been given $\int(2x-2)\cos(x^2-2x)dx$ without the previous exercise, it certainly would have been hard to recognize. We develop a technique that will make this easier, called **u-substitution**.

Technique to anti-diff. chain rule

1.2 Indefinite Integrals With Substitution

$$u = g(x)$$
$$\frac{du}{dx} = g'(x)$$
$$\boxed{du = g'(x)dx}$$

Theorem 1.2 (The Substitution Rule) *If $u = g(x)$ is differentiable, then:*

$$\boxed{\int g'(x) f(\underbrace{g(x)}_{u})dx = \int f(u)du}$$

The procedure is best summed up in a few steps, alongside an example:

Example 1.3

$$\int \underbrace{e^{\tan(x)}}_{f(g(x))} \sec^2(x)dx$$

1. *In your integral, identify a factor that looks like a composition of two functions. Which is the inner function? Which is the outer function?*

   inner: $\tan(x) = g(x)$

   outer: $e^x = f(x)$

2. *Do you see the derivative of the inner function elsewhere in the function? We are looking for an expression of the form $f(g(x)) \cdot g'(x)$.*

   $g'(x)$ = deriv. of inner function $= \sec^2(x)$

3. *Let $u = g(x)$.* Let $\boxed{u = \tan(x)}$

4. *Find $\frac{du}{dx}$:* $\frac{du}{dx} = \sec^2(x)$ Take deriv

   solve for du.

5. *Find an expression for dx in terms of du:* $du = \sec^2(x)dx$ $\Big\}$ $dx = \frac{du}{\sec^2(x)}$

6. *Replace everything in the integral that has to do with x with expressions involving u. In particular, $u = g(x)$ and $du = g'(x)dx$.*

   $\int e^{\tan(x)} \sec^2(x)dx = \int e^u du$ NO More x's!!!

   $\int e^u \cancel{\sec^2(x)} \cdot \frac{du}{\cancel{\sec^2(x)}} = \int e^u du$

7. *Evaluate the integral with respect to u*

   $\int e^u du = e^u + C$

8. *Un-do the replacement, and replace u with $g(x)$.* plug back in

   $\int e^{\tan(x)} \sec^2(x)dx = \boxed{e^{\tan(x)} + C}$

   Check:

   $\frac{d}{dx}\left(e^{\tan x} + C\right) =$

   $e^{\tan(x)} \cdot \sec^2(x) + 0$

   $e^{\tan(x)} \sec^2(x)$ ✓

Example 1.4

$$\int (x^4-1)^{10} x^3\,dx$$

1. *In your integral, identify a factor that looks like a composition of two functions. Which is the inner function? Which is the outer function?*

inner: $x^4-1 = g(x)$

outer: $x^{10} = f(x)$ $\quad f(g(x)) = (x^4-1)^{10}$

2. *Do you see the derivative of the inner function elsewhere in the function? We are looking for an expression of the form $f(g(x)) \cdot g'(x)$.*

x^4-1

deriv. of inner function: $\frac{d}{dx}(x^4-1) = 4x^3$

3. *Let $u = g(x)$.* $\quad u = x^4-1$

4. *Find $\frac{du}{dx}$:* $\quad \frac{du}{dx} = 4x^3$

5. *Find an expression for dx in terms of du:*
(or du in terms of x)

$du = 4x^3dx$

$\frac{1}{4}du = x^3dx$

$dx = \frac{du}{4x^3}$

6. *Replace everything in the integral that has to do with x with expressions involving u. In particular, $u = g(x)$ and $du = g'(x)dx$.*

$$\int (x^4-1)^{10} x^3dx = \int (u)^{10} \cdot \frac{1}{4}du = \frac{1}{4}\int u^{10}du$$

7. *Evaluate the integral with respect to u.*

$$\frac{1}{4}\int u^{10}du = \frac{1}{4}\left(\frac{u^{11}}{11} + C\right) = \frac{1}{44}u^{11} + \underbrace{\frac{1}{4}C}_{\text{still any constant}}$$

8. *Un-do the replacement, and replace u with $g(x)$.*

$$\frac{1}{44}\cdot u^{11} + K \rightsquigarrow \boxed{\frac{1}{44}(x^4-1)^{11} + K}$$

$\frac{1}{4}\left(\frac{(x^4-1)^{11}}{11} + C\right)$

Example 1.5

$$\int x\cos(x^2)dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int x\cos(x^2)dx = \int \cos(u)\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\int \cos(u)du$$

$$= \frac{1}{2}(\sin(u) + C)$$

$$= \boxed{\frac{1}{2}(\sin(x^2) + C)}$$

1.3 Definite Integrals With Substitution

With definite integrals, we can use the same process but we need to be careful about the bounds: remember that they define x-values, not u-values, so you want to return to x before using the evaluation theorem:

Example 1.6

$$\int_1^2 \frac{\ln(x)}{x}dx = \int_{x=1}^{x=2} \frac{\ln(x)}{x}dx$$

never gets us anywhere

$u = x$

$du = dx$

Example 1.7

$$\int_{-1}^{1} \frac{e^x}{e^x - 5}dx$$

1.4 Symmetry

Suppose f is continuous on $[-a, a]$.

(a) If f is even (so $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)dx =$$

(b) If f is odd (so $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)dx =$$

1.4.1 Common Even Functions

(Don't forget trig functions!)

1.4.2 Common Odd Functions

1.4.3 Examples

Example 1.8

$$\int_{-3}^{3} \frac{\sin(x)}{x^4 + 2x^2 + 3} dx$$