05.03

Monday, November 16, 2020 12:37 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 5.3: Evaluating Definite Integrals

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Today's Goal: A new technique to evaluate definite integrals.

Logistics: We will start this on [illegible]. There is a check-in on Friday (always!), and we will start section 5.4 on Friday as well.

$-\int_0^4 \sqrt{16-x^2}\,dx$

Warm-Up 1.1 *Using geometry, evaluate the integral* $\int_4^0 \sqrt{16-x^2}\,dx$.

$y^2 + x^2 = 16$

$y^2 = 16 - x^2$

$y = \sqrt{16-x^2}$ pos. sq. root, upper-half of circle

(A) 2π

(B) 4π

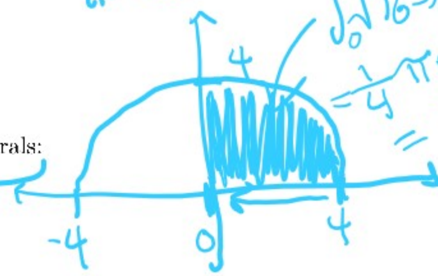
(C) 8π

(D) 16π

(E) *None of the above*

$$\int_4^0 \sqrt{16-x^2}\,dx = -4\pi$$

$$\int_0^4 \sqrt{16-x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2 = 4\pi$$



So far, we have two methods for evaluating deifinite integrals: finding area under a curve

1. Geometry
2. Taking a limit of a Riemann sum

Today, we will learn another technique, and this technique ties areas under curves back to antiderivatives that we had practiced in Section 5.1.

← 2 parts

This method is part of the Fundamental Theorem of Calculus, which we will talk about more in 5.4. It is sometimes known as the "Evaluation Theorem".

Theorem 1.2 (Fundamental Theorem of Calculus Part II) *Suppose f is continuous on $[a, b]$ and F is any antiderivative of f (i.e., $F' = f$). Then,*

just need one!

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 1.3 *Use the evaluation theorem to evaluate the integral:*

$$\int_2^3 2^x dx$$

$f(x) = 2^x$

$F'(x) = 2^x$

$\boxed{\frac{d}{dx}\left(\frac{1}{\ln(2)}2^x\right) = 2^x?}$

simplest anti-deriv. of 2^x to work with (+C not needed)

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(2^x) = \ln(2)2^x$

$F(x) = \frac{1}{\ln(2)}2^x$

$$\int_2^3 2^x dx = F(3) - F(2) = \frac{1}{\ln(2)}2^3 - \frac{1}{\ln(2)}2^2 = \frac{8}{\ln(2)} - \frac{4}{\ln(2)} = \frac{4}{\ln(2)}$$

1.0.1 Jusitfying the Evaluation Theorem

"Evaluation Theorem"

Theorem 1.4 (Fundamental Theorem of Calculus Part II) *Suppose f is continuous on $[a, b]$ and F is any antiderivative of f (i.e., $F' = f$). Then,*

$$\int_a^b f(x)dx = F(b) - F(a).$$

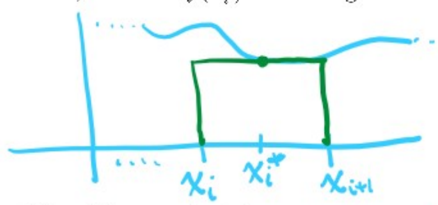
Recall that the definite integral is defined as a limit of a Riemann sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\overbrace{f(x_i^*)}^{\text{height}}\overbrace{\Delta x}^{\text{width of rectangle} = \frac{b-a}{n}}}_{\text{Area of Rect.}}$$

Lets consider one of these infinitely small intervals in particular. We will have to use a stretch of imagination, as *infinitely small* width is not easy to imagine.

But let's suppose $[x_i, x_{i+1}]$ is an infinitely small interval, and that $f(x_i^*)$ is the height of our infinitely small rectangle.

i^{th} rectangle



Recall that $f = F'$, and consider the Intermediate Value Theorem in this context:

$$\boxed{\frac{F(x_{i-1}) - F(x_i)}{x_{i+1} - x_i}} = F'(c) = f(c)$$

slope of a secant line

slope of a tangent line for some c in $[x_i, x_{i+1}]$

for some c in $[x_i, x_{i-1}]$.

What if our c was x_i^*? Then the IVT tells us:

$$\left[\frac{F(x_{i+1}) - F(x_i)}{\underbrace{x_{i+1} - x_i}_{\text{width of interval}}} = f(x_i^*)\right]$$

The width of our interval is Δx:

$$\frac{F(x_{i+1}) - F(x_i)}{\Delta x} = f(x_i^*) \quad F' = f$$

And if we move the Δx to the other side we have:

$$\underbrace{F(x_{i+1}) - F(x_i)}_{} = \underbrace{f(x_i^*)\Delta x}_{\text{Area of Rectangle } i}$$

Putting these all back in our Riemann sum gives:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)\Delta x}_{\text{By IVT}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(F(x_{i+1}) - F(x_i)\right)$$

$$= F(b) - F(b-\cancel{\Delta x}) + F(b\cancel{-\Delta x}) - F(b\cancel{-2\Delta x}) \cdots - F(a)$$

$$= F(b) - F(a)$$

Example 1.5 *Use the Evaluation Theorem to evaluate the following integrals:*

(a) $\int_0^{\pi/4} \sin(x)dx$

(b) $\int_{-1}^{1} e^x dx$

(c) $\int_0^{\pi/3} \sec(t)\tan(t)dt$

Area under $y = \sec(t)\tan(t)$ between $x = 0$ and $x = \pi/3$

$= \sec(\pi/3) - \sec(0)$

$= \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(0)} = \frac{1}{1/2} - \frac{1}{1} = 2 - 1 = 1$

$\tan(\_\_) = -1$?

$(-\pi/2, \pi/2)$

(d) $\int_{-1}^{0} \frac{1}{1+y^2} dy$

$=$

$\arctan(0) - \arctan(-1)$

$0 - -\pi/4 = \pi/4$

1.1 Indefinite Integrals

In the past, we have simply asked for the family of antiderivatives of a particular function: No notation.

Example 1.6 *Find all possible antiderivatives of the function* $G(x) = x^3 - e^x$.

same question! new notation

$$\boxed{\frac{x^4}{4} - e^x + C}$$

Now that we have the evaluation theorem and the notation of integrals, we can rephrase this question using a new notation for antiderivatives: the **indefinite integral**:

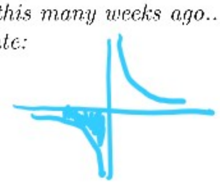
Example 1.7 *Find* $\int (x^3 - e^x)dx$ ← the family of all possible anti-deriv.'s of $(x^3 - e^x)$

$$\boxed{\int (x^3 - e^x)\,dx = \frac{x^4}{4} - e^x + C}$$

★ Indef. integrals are families of functions

★ Definite integrals (w/ bounds) are numbers

Example 1.8 *This is an important example, and has a bit of a twist! Try to remember a discussion we had about this many weeks ago...*
Evaluate:



$$\int \frac{1}{x}dx$$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

domain$(\ln(x)) = (0, \infty)$

$$★ \boxed{\int \frac{1}{x}dx = \ln|x| + C} ★$$

Example 1.9

$$\int \frac{2\sqrt{x} - 3x^6 + 2\sqrt[3]{x^2} - \pi}{x^2}dx \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\frac{2x^{1/2}}{x^2} - \frac{3x^6}{x^2} + \frac{2x^{2/3}}{x^2} - \frac{\pi}{x^2} \to \int (2x^{-3/2} - 3x^4 + 2x^{-4/3} - \pi x^{-2})\,dx$$

$$= \frac{2x^{-1/2}}{-1/2} - \frac{3x^5}{5} + \frac{2x^{-1/3}}{-1/3} - \frac{\pi x^{-1}}{-1} + C$$

$$= -4x^{-1/2} - \frac{3}{5}x^5 - 6x^{-1/3} + \pi x^{-1} + C$$

$$= \boxed{-\frac{4}{\sqrt{x}} - \frac{3}{5}x^5 - \frac{6}{\sqrt[3]{x}} + \frac{\pi}{x} + C}$$

1.2 Applications

Evaluation theorem

Theorem 1.10 (Net Change Theorem) *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

net change in F between $x = a$ and $x = b$

If a particle moves from position 10 to position 19, that is a net change of 9

For example, we know that velocity is the rate of change of position. This tells us:

$$\underbrace{\int_a^b v(t)dt}_{\text{net change in position!}} = \overbrace{s(b) - s(a)}^{\text{net change in position}},$$

where s is position and v is velocity, so $s' = v$.

Example 1.11 *A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)dt$ represent?*

$n'(t)$ = rate of change in population

$\int_0^{15} n'(t)dt$ = net change in n between $t=0$ and $t=15$

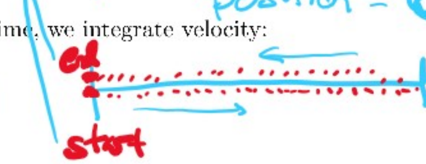
n = # of bees in population

Since 100 is the starting population, the expression $100 + \int_0^{15} n'(t)dt$ gives the # of bees in the pop. at $t = 15$.

1.2.1 Displacement vs. Distance

start and end at the same spot, so net change in position = 0

If we want to calculate the displacement of a moving object over time, we integrate velocity:



$$\int_a^b v(t)dt = s(b) - s(a),$$

so $s(b) - s(a)$ gives the net change in position.

But what if we want to know the **total distance** traveled by the object? In other words, what if we want to count all motion as positive distance?

$$\text{total distance traveled} = \int_a^b |v(t)|dt$$

make the neg. velocity positive using the abs. value.

Example 1.12 *The velocity function is given by $v(t) = t^2 - 2t$ for a particle moving along a line. Find both the displacement and the distance traveled by the particle during the time interval $1 \le t \le 6$.*

Displacement = net change in position

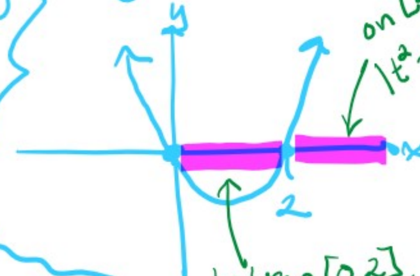
$$= \int_1^6 (t^2 - 2t)\,dt = \frac{t^3}{3} - t^2 \Big|_1^6 = \frac{6^3}{3} - 6^2 - \left(\frac{1^3}{3} - 1^2\right)$$

$$= \frac{216}{3} - 36 - \frac{1}{3} + 1 \quad \text{.... calculator}$$

$$= 72 - 36 - \frac{1}{3} + 1$$

$$= 37 - \frac{1}{3} = \boxed{36\tfrac{2}{3}}$$

$t^2 - 2t$



on $[2, \infty)$, $|t^2 - 2t| = t^2 - 2t$

between $[0, 2]$, $|t^2 - 2t| = -(t^2 - 2t)$

Total distance $= \int_1^6 |t^2 - 2t|\,dt$

$$= \int_1^2 \left(-(t^2 - 2t)\right)dt + \int_2^6 (t^2 - 2t)\,dt$$

$$= \int_1^2 (-t^2 + 2t)\,dt + \int_2^6 (t^2 - 2t)\,dt = -\frac{t^3}{3} + t^2 \Big|_1^2 + \left(\frac{t^3}{3} - t^2\right)\Big|_2^6$$

$$= \left(-\frac{8}{3} + 4\right) - \left(-\frac{1}{3} + 1\right) + \left(\frac{6^3}{3} - 6^2\right) - \left(\frac{8}{3} - 4\right) = -5 + 3 + 36 + 4$$

$$= 38$$

Example 1.13 *Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 minutes.*

★ Be able to write $|t^2 - 2t + 1|$ as a piecewise function!

→ "Abs value functions as piecewise functions"

$$= \int_0^{10} (r(t))\,dt$$

$$= \int_0^{10} (200 - 4t)\,dt = 200t - 2t^2 \Big|_0^{10} = 200(10) - 2(10)^2 - 0$$

$$= 2000 - 2 \cdot 100 = 1800 \text{ liters}$$