05.02 Definite Integral

Wednesday, November 11, 2020 12:35 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 5.2: Definite Integrals

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Today's Goal: Define the definite integral; relate to the area problems we've been working on.

Logistics: We will start this on Wednesday and need to finish it on the following Monday. There is an activity on Friday. And a check-in on ~~MONDAY~~ Friday this time around! As well as the following Friday (so Monday and Friday of the same week).

Warm-Up 1.1 *What is the area under the curve $y = x + 1$ between $x = 0$ and $x = 2$? Hint: Draw a picture to help - you can use geometric area formulas!*

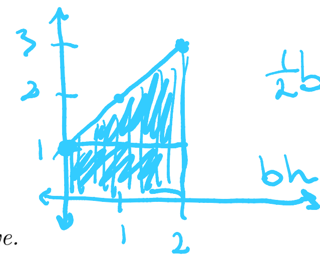
(A) 1

(B) 2

(C) $\frac{3}{2}$

(D) 4

(E) None of the above.



$\frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2$

$bh = 2 \cdot 1$

4

Last section, we used rectangles to approximate areas, and limits to find them precisely. We will do a little review of this by discussing Midpoint Rule. Then, we will introduce a new notation for these precise areas, discuss when this object exists, and develop a new way to find these values.

1.1 Midpoint Rule

When we are computing sums of rectangle areas to estimate an area under the curve (computing a **Riemann Sum**, we have many options for choosing the height of the curve. Midpoint rule describes one such choice: using the midpoints of the intervals defining the rectangle widths.

\# of rectangles

Example 1.2 *Use midpoint rule with $n = 5$ to approximate the area under the curve $y = \frac{2}{x+1}$ between $x = 1$ and $x = 3$.*

No limit :)

$1 \quad x_1^* \quad \frac{7}{5} \quad x_2^* \quad \frac{9}{5} \quad x_3^* \quad \frac{11}{5} \quad x_4^* \quad \frac{13}{5} \quad x_5^* \quad 3$

$= \frac{5}{5} \quad 1.4 \quad 1.8 \quad 2.2 \quad 2.6 \quad = \frac{15}{5}$

5 rectangles: width $= \frac{3-1}{5} = \frac{2}{5}$

To find rectangle heights, find the midpoints of the intervals above and plug into $f(x) = \frac{2}{x+1}$.

Rectangle	Midpoint	Height
1	1.2	$\frac{2}{2.2}$
2	1.6	$\frac{2}{2.6}$
3	2	$\frac{2}{3}$
4	2.4	$\frac{2}{3.4}$
5	2.8	$\frac{2}{3.8}$

$\text{Area} = \sum_{i=1}^{5} f(x_i^*)(\Delta x)$

$\Delta x = .4$

$\boxed{\text{Area} = .4\left(\frac{2}{2.2} + \frac{2}{2.6} + \frac{2}{3} + \frac{2}{3.4} + \frac{2}{3.8}\right)}$

Area = (calculator)

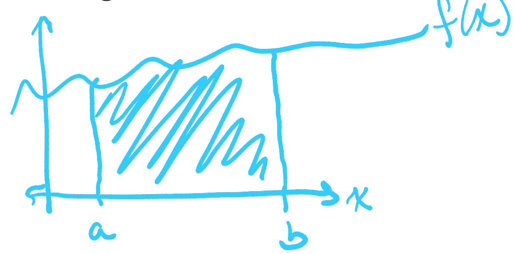
1.2 New Notation for Area

The area under the curve $f(x)$ on the interval $[a, b]$ is computed by a limit of a sum of rectangle areas (a **Riemann sum**):

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}, \quad \Delta x = \frac{b-a}{n}$$

We introduce a new notation, using a symbol called an **integral:**

$$\int_a^b \underbrace{f(x)}_{\text{"}f(x_i^*)\text{"}} \underbrace{dx}_{\text{"}\Delta x\text{"}}$$



1.3 Evaluating Integrals as Limits of Riemann Sums

Example 1.3 *Evaluate* $\int_0^3 (x^3 - 2x)dx$ *using a limit of Riemann sums.*

Example 1.4 *Set up an expression for* $\int_1^2 2^x dx$ *as a limit of sums. Do not evaluate.*

$[a,b] = [1,2]$

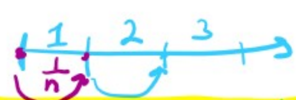
$f(x) = 2^x$

the area under $f(x) = 2^x$ between $x = 1$ and $x = 2$

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x\right)$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

Rect	Height
1	$f(1) = 2^1$
2	$f(1+\frac{1}{n}) = 2^{1+\frac{1}{n}}$
3	$f(1+\frac{2}{n}) = 2^{1+\frac{2}{n}}$
$\vdots$	
i	$f(1+\frac{(i-1)}{n}) = 2^{1+\frac{i-1}{n}}$
$\vdots$	
n	$f(1+\frac{(n-1)}{n}) = 2^{1+\frac{n-1}{n}}$



$$\boxed{A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} 2^{1+\frac{i-1}{n}}\cdot \frac{1}{n}\right)}$$

We can also use the geometric interpretation of an integral in order to evaluate that integral. A few common examples of this:

area under $y=\sqrt{4-x^2}$ between $x=-2, x=2$

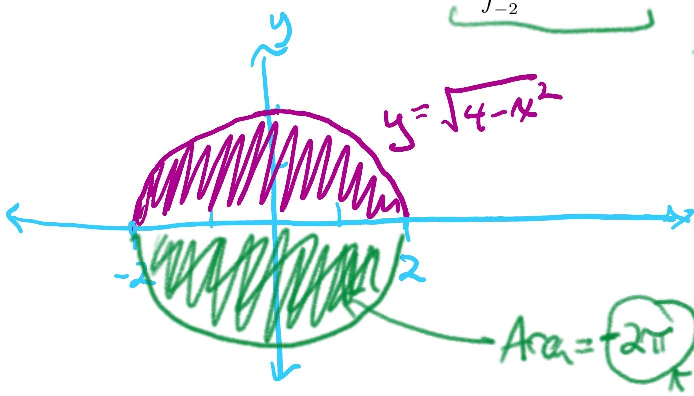
Example 1.5 *Use geometry to evaluate:*

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$y=\sqrt{4-x^2}$ ← pos. y-vals.

$y^2 = 4-x^2$

$x^2+y^2=4$ circle rad. 2



$y=-\sqrt{4-x^2}$

bottom half of the circle

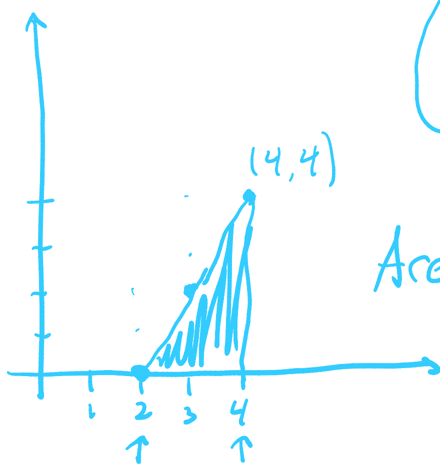
Area $=-2\pi$ ← neg. b/c below

Area of circle $=\pi r^2$

Area of semicircle $=\frac{1}{2}\pi r^2 = 2\pi$ ← above x-axis

Example 1.6 *Express the area under the curve $y = 2x - 4$ between $x = 2$ and $x = 4$ as an integral. Draw this region, and evaluate the integral geometrically.*

$$\int_2^4 (2x-4)\,dx = 4$$



Area $=\frac{1}{2}bh$

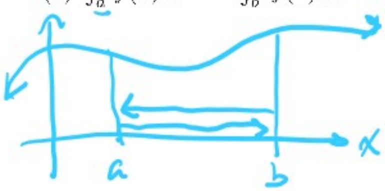
$= 4$

$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$

$\int_a^b f(x)dx$ = The area under the graph of f(x) between $x=a$ and $x=b$

1.4 Properties of the integral

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

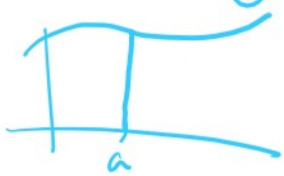


$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\underbrace{\Delta x}_{\frac{b-a}{n}}$

$\int_b^a f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

this time we have $\frac{a-b}{n} = \frac{-(b-a)}{n}$

(2) $\int_a^a f(x)dx = 0$

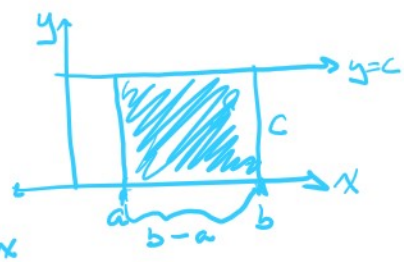


$\int_{e^\pi}^{e^\pi} \sin^2(x^4-1)dx = 0$

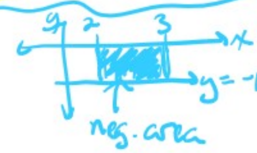
(3) $\int_a^b c\,dx = c(b-a)$

c is any real #

$y = c$

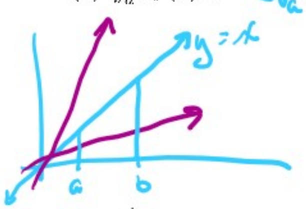


$\int_{-1}^{2} \pi\, dx = 3\pi$



$\int_2^3 (-1)dx = -1$

(4) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

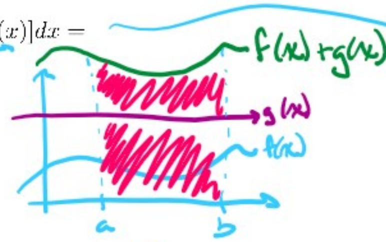


A = height · width

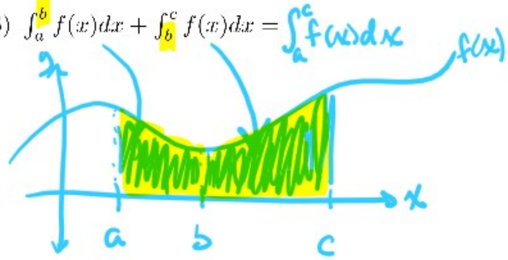
If height is multiplied by c, then so is A.

(5) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

square brackets



(6) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$



$$\boxed{\begin{aligned}\int_{-1}^{2} f(x)dx &= 3\\ \int_{4}^{2} f(x)dx &= -7\end{aligned}}$$

$\int_2^4 f(x)dx = 7$

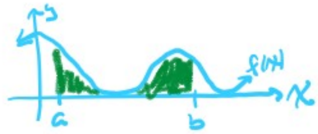
Find $\int_{-1}^{4} f(x)dx = \int_{-1}^{2} f(x)dx + \int_{2}^{4} f(x)dx$

$= 3 + 7 = 10$

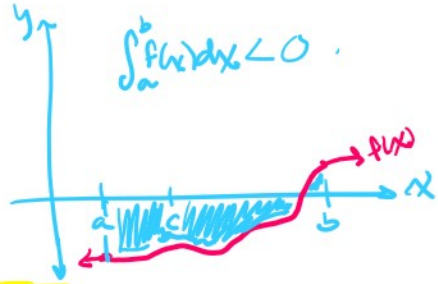
Positive area

(7) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

f(x) is above x-axis

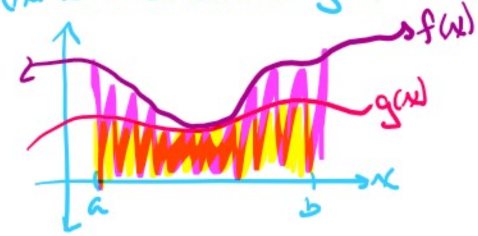


★ Is it true that if $f(c) < 0$ for some c: $a \leq c \leq b$ that $\int_a^b f(x)dx < 0$?



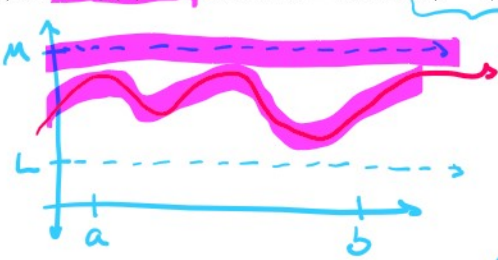
(8) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

f(x) is = or above g(x)



$\overbrace{\text{heights}}$

(9) If $L \leq f(x) \leq M$ for $a \leq x \leq b$, then $L(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.



$$L(b-a) \leq \int_a^b f(x)dx$$
$$\int_a^b L\,dx \leq \int_a^b f(x)dx$$

$$\int_a^b f(x)dx \leq \underbrace{M(b-a)}_{\int_a^b M\,dx}$$

Example:

$$\int_2^5 f(x)dx = 9$$

Find $\int_2^5 (2\pi - 3f(x))dx$.

$$= \int_2^5 2\pi\,dx - \int_2^5 3f(x)dx$$
$$= (5-2)2\pi - 3\int_2^5 f(x)dx$$
$$= 6\pi - 3(9) = \boxed{6\pi - 27}$$