4.5 Indeterminate Forms and l'Hopital's Rule

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Math 1300: Calculus I **Fall 2020**

Lecture: Section 4.5: Indeterminate Forms and l'Hôpital's Rule

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Today's Goal: Learn about more complicated limits and how Calculus can help.

Logistics: We will start this on Monday, finish on Wednesday. Tuesday we have an activity and can do some light quiz review. Bring your questions for Tuesday! Quiz 5 is TUESDAY NIGHT (open at 7pm - set an alarm!). It covers: 3.7, 3.8, 3.9, 4.2, 4.3, Activities from 10/13 & 10/20 (Projects 8 & 9).

Friday we have another activity.

$y = m(x - x_1) + y_1$

$f(x) = (1 - x^2)^{1/3}$

Warm-Up 1.1 *Find the linearization of* $f(x) = \sqrt[3]{1 - x^2}$ *at* $x = 3$.

$f(3) = \sqrt[3]{-8} = -2$

point: $(3, -2)$

(A) $L(x) = \frac{1}{2}(x - 3) + 2$

(B) $L(x) = \frac{-1}{2}(x - 3) - 2$

(C) $L(x) = \frac{-1}{2}(x - 3) + 2$

(D) $L(x) = \frac{1}{2}(x - 3) - 2$

(E) *None of the above.*

$f'(x) = \frac{1}{3}(1 - x^2)^{-2/3} \cdot -2x$

$f'(x) = \dfrac{-2x}{3(1 - x^2)^{2/3}}$

$f'(3) = \dfrac{-6}{3(-8)^{2/3}} = \dfrac{-6}{3(\sqrt[3]{-8})^2} = \dfrac{-6}{3 \cdot 4} = \dfrac{-1}{2}$

slope

1.1 Remember Limits?

Example 1.2 *Find the following limit:*

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 + 2x - 3}$$

1) Try plugging in $x = 1$: $\dfrac{1^2 - 1}{1^2 + 2 - 3} = \dfrac{0}{0}$ ... need more techniques.

2) Factor: $\displaystyle \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}} = \lim_{x \to 1} \frac{x+1}{x+3}$

$= \dfrac{2}{4} = \dfrac{1}{2}$ ✓

1.1.1 What can go wrong?

Example 1.3 *Find the limit:*

$$\lim_{x\to 1} \frac{\ln(x)}{x-1}$$

1) Plug in $x=1$: $\frac{0}{0}$ need more techniques

This time we can't fix it by factoring. However, we can use another clever trick. Remember how we were using the linearization to approximate the function? This works for functions in limits too. In particular, when:

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \frac{0}{0},$$

We can plug in the linearizations of f and g:

linearization of $f(x)$ at $x=1$

$$\lim_{x\to 1} \frac{f(x)}{g(x)} \approx \lim_{x\to 1} \frac{f'(1)\cancel{(x-1)} + \cancel{f(1)}}{g'(1)\cancel{(x-1)} + \cancel{g(1)}} \approx \frac{f'(1)}{g'(1)}$$

linearization of $g(x)$ at $x=1$

The above is a rough idea (a real proof would need more rigorous analysis than we have at this point), but it gives us a theorem:

Theorem 1.4 (L'Hôpital's Rule) *Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty},$$

then:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right side exists or is $\pm\infty$.

Now let's finish that example!

1) $\lim_{x\to 1} \frac{\ln(x)}{x-1} \to \frac{0}{0}$ indet. form.

2) By l'Hôpital's rule:

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1}$$

$$= \lim_{x\to 1} \frac{1}{x} = \boxed{1}$$

Ex: $\lim_{x\to 0} \sin(x)\cdot\left(\frac{1}{x}\right) \Rightarrow 0\cdot\infty$

$\lim_{x\to 0} \frac{\sin(x)}{x} \Rightarrow \frac{0}{0}$

l'Hôp. applies: $= \lim_{x\to 0} \frac{\cos(x)}{1} = 1$

1.1.2 What are "indeterminate forms"?

We've just seen an example: $\frac{0}{0}$, and talked about $\frac{\infty}{\infty}$.

1. Are there others?
2. How do we get them to look like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use L'Hôpital's rule?

Wikipedia has an excellent list of them!
`https://en.wikipedia.org/wiki/Indeterminate_form#List_of_indeterminate_forms`

use l'Hôpital's Rule

$\lim_{x\to 0} \sin(x)\cdot\frac{1}{x}$

Indeterminate form	Conditions	Transformation to $0/0$	Transformation to ∞/∞
$\frac{0}{0}$	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=0$	—	$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
$\frac{\infty}{\infty}$	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	—
$\boxed{0\cdot\infty}$	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c}\frac{g(x)}{1/f(x)}$
$\infty-\infty$	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x)) = \ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x)=0^+, \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\boxed{\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}}$
1^∞	$\lim_{x\to c} f(x)=1,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$

Example 1.5 *Evaluate* $\lim_{x\to 0^+} x\ln(x)$. $\rightsquigarrow$ "$0\cdot\ln(0)$" $\rightsquigarrow$ "$0\cdot(-\infty)$"

This one does not look like $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Is there anything we can do to get it there? Keep in mind we will be taking derivatives.

① to get this to look like $\frac{0}{0}$: $\lim_{x\to 0^+}\frac{x}{1/\ln(x)} \rightsquigarrow \frac{0}{1/\infty} \rightsquigarrow \frac{0}{0}$

② to get this to look like $\frac{\pm\infty}{\infty}$: $\lim_{x\to 0^+}\frac{\ln(x)}{1/x} \rightsquigarrow \frac{-\infty}{\infty}$

② is easier to take derivs: By l'H: $= \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim -\frac{x^2}{x} = \lim(-x) = \boxed{0}$

$\ln|x|$:

Example 1.6 *Find the limit* $\lim_{x\to 0^+}(\tan(2x))^x$. $\rightsquigarrow$ 0^0 indet. form!

$\lim_{x\to 0^+}(\tan(2x)^x) = y$

$$\ln(y) = \lim_{x\to 0^+} \ln(\tan(2x)^x)$$

$$\ln(y) = \lim_{x\to 0^+} x \cdot \ln(\tan(2x))$$

$$\ln(y) = \lim_{x\to 0^+} \frac{\ln(\tan(2x))}{1/x} \to \frac{-\infty}{\infty}$$

By l'Hôpital's rule:

$$\ln(y) = \lim_{x\to 0^+} \frac{\frac{1}{\tan(2x)}\cdot \sec^2(2x)\cdot 2}{-1/x^2}$$

$$\ln(y) = \lim_{x\to 0^+} \frac{-2x^2\sec^2(2x)}{\tan(2x)} \to \frac{0}{0}$$

By l'Hôpital's Rule:

$$\ln(y) = \lim_{x\to 0^+} \frac{-\cancel{4}^{2}x\cancel{\sec^2(2x)} - 2x^2\cdot 2\cancel{\sec(2x)}\cdot \sec(2x)\tan(2x)\cdot \cancel{2}}{\cancel{\sec^2(2x)}\cdot \cancel{2}}$$

$$\ln(y) = \lim_{x\to 0^+} -2x - 4x^2\tan(2x)$$

$$\ln(y) = 0$$

$$y = e^0 = 1$$

$$\boxed{\lim_{x\to 0^+}(\tan(2x)^x) = 1}$$

Example 1.7 *Evaluate the limit:* $\lim_{x\to\infty} x^2e^x$

this limit goes to $\boxed{\infty}$ (not an indet. form.)

Example 1.8 *Evaluate the limit:* $\lim_{x\to-\infty} x^2e^x$

"$(-\infty)^2\cdot e^{-\infty}$" $\rightarrow$ "$\infty\cdot 0$"

$\lim_{x\to-\infty}\left(\frac{e^x}{(1/x^2)}\right) \rightsquigarrow$ "$\frac{0}{0}$"

By l'Hôp: "$\lim_{x\to-\infty}\frac{e^x}{-2x^{-3}}$" $\rightsquigarrow$ "$\frac{0}{0}$"

harder, try other option:

$= \lim_{x\to-\infty}\frac{x^2}{e^{-x}} \to \frac{\infty}{\infty}$

By l'Hôpital's:

$= \lim_{x\to-\infty}\frac{2x}{-e^{-x}}$

still get $\frac{-\infty}{-\infty}$, so use l'Hôpital's:

$$= \lim_{x\to-\infty}\frac{2}{e^{-x}} \rightsquigarrow \frac{2}{\infty} = \boxed{0}$$

Example 1.9 *Find the limit* $\lim_{x\to 0}\frac{\sin^2(x)}{x}$.

looks like $\frac{0}{0}$. Apply l'Hôpital's:

$$= \lim_{x\to 0}\frac{2\sin(x)\cos(x)}{1} = \boxed{0}$$

Example 1.10 *Find the limit* $\lim_{x\to 0}(1-2x)^{1/x}$.

looks like 1^∞, need logs:

$$y = \lim_{x\to 0}(1-2x)^{1/x}$$

$$\ln(y) = \lim_{x\to 0}\frac{1}{x}\cdot\ln(1-2x)$$

$$= \lim_{x\to 0}\frac{\ln(1-2x)}{x} \text{ looks like } \frac{0}{0}$$

By l'Hôpital's:

$$\ln(y) = \lim_{x\to 0}\frac{\frac{1}{1-2x}\cdot -2}{1}$$

$$\ln(y) = \lim_{x\to 0}\frac{-2}{1-2x} = \boxed{-2}$$