04.03 MVT and Curve Sketching

Tuesday, October 20, 2020 12:38 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 4.3: Derivatives and Curve Sketching

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Today's Goal: Discuss the mean value theorem and how to sketch the graph of any function.

Logistics: We will start this section with an activity Tuesday and finish Wednesday. There is a check-in Friday, it's set to cover the topics we are covering this week: linear approximation, min/max problems, and extreme value theorem (Sections 3.9 & 4.2).

Warm-Up 1.1 *What are the critical numbers of the function $f(x) = x^3 - 12x$?* SKIP

(A) $-2, 0, 2$

(B) $-4, 4$

(C) $-2, 2$

(D) $-4, 0, 4$

(E) None of the above.

skip this one

$f'(x) = 3x^2 - 12$

$= 3(x^2 - 4) = 3(x+2)(x-2)$

Do this one

Warm-Up 1.2 *Use logarithmic differentiation to find y' for $y = (\sin(x))^{x^2}$.*

(A) $y' = (\sin x)^{x^2}(\frac{2x\cos(x)}{\sin x})$

(B) $y' = -(\sin x)^{x^2}(\frac{2x\cos(x)}{\sin x})$

(C) $y' = (\sin x)^{x^2}(2x\ln(\sin x) - \frac{x^2\cos(x)}{\sin(x)})$

(D) $y' = (\sin x)^{x^2}(2x\ln(\sin x) + \frac{x^2\cos(x)}{\sin(x)})$

(E) None of the above

$\ln(y) = \ln((\sin x)^{x^2})$

$\ln(y) = \underline{x^2} \cdot \underline{\ln(\sin x)}$ Apply $\frac{d}{dx}$ both sides

* y is $y(x)$!

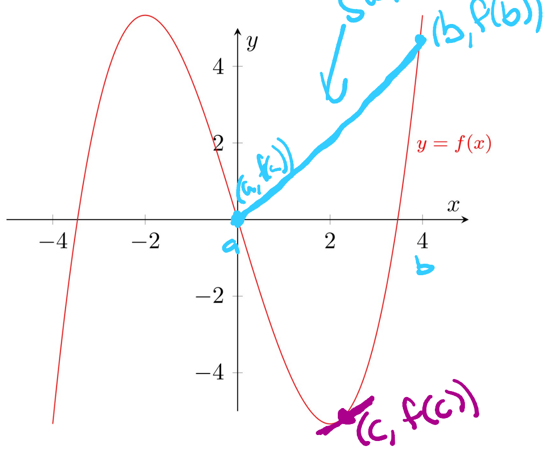
$\frac{1}{y} \cdot y' = 2x\ln(\sin(x)) + x^2 \cdot \frac{1}{\sin(x)} \cdot \cos(x)$

$y' = y\left(2x\ln(\sin(x)) + \frac{x^2\cos(x)}{\sin(x)}\right)$

$y' = (\sin x)^{x^2}\left(2x\ln(\sin x) + \frac{x^2\cos(x)}{\sin(x)}\right)$

1.1 The Mean Value Theorem

slope = $\frac{f(b) - f(a)}{b - a}$



★ The slope of this secant line must be achieved by some tangent line for a point in between.

Theorem 1.3 (The Mean Value Theorem) *If is a differentiable function on the interval* $[a, b]$*, then there exists a number c in* (a, b) *such that*

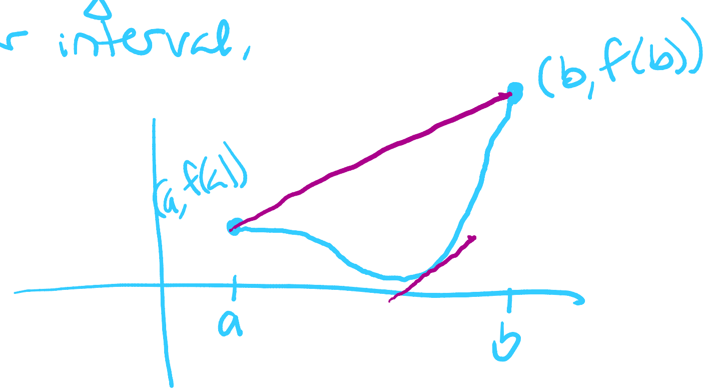
the slope of the tangent lines at (c, f(c))

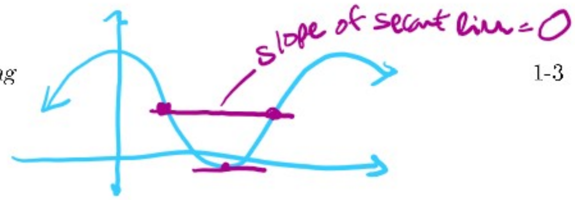
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

this is slope of the secant line connecting (a, f(a)) and (b, f(b))

- Draw a secant line betwen endpoints of interval.
- Find the slope

→ The MVT guarantees that there will be a tangent line that has that slope, in your interval.





Remark 1.4 *When the slope of the secant line is equal to 0, this is called* **Rolle's Theorem**.

(just a special case MVT)

Example 1.5 *Use the Mean Value Theorem to show that $f'(c) = 0$ for some $c \in (-1, 3)$, where $f(x) = x^2 - 2x - 8$.*

Hyp. NTS: f is diff on the closed interval $[-1, 3]$
This true b/c f is a polynomial, so f is diff everywhere $(-\infty, \infty)$.

$3^2 - 2(3) - 8$
$9 - 6 - 8 = -5$
$(-1)^2 + 2 - 8 = -5$

Slope of secant: ★ Need: slope of secant line to be 0:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(-1)}{3 - -1}$$

$$= \frac{-5 - -5}{4} = 0$$

conclusion

By the MVT, there exists $c \in (-1, 3]$ w/ $f'(c) = 0$.

Example 1.6 *Suppose we know that $f(x)$ is continuous and differentiable on $[-7, 0]$ and that $f(-7) = -3$ and that $f'(x) \leq 2$. What is the largest possible value for $f(0)$?*

hypotheses of MVT

By the MVT, there exists c in $(-7, 0)$ st:

$$f'(c) = \frac{f(0) - f(-7)}{0 - -7}.$$

We also know $f'(c) \leq 2$.

So $$\frac{f(0) - f(-7)}{0 - -7} \leq 2$$

$$\frac{f(0) - f(-7)}{7} \leq 2$$

$$f(0) - f(-7) \leq 14$$

$$f(0) - -3 \leq 14$$

$\rightarrow f(0) \leq 11$

★ 11 is the largest possible value of $f(0)$.

1.2 Sketching Techniques

1.2.1 Increasing/Decreasing

- If $f'(x) > 0$ on an interval, then... f is inc. on that interval
- If $f'(x) < 0$ on an interval, then... f is dec. on that interval

1.2.2 Local Minima and Maxima

Theorem 1.7 **(First Derivative Test)** *Suppose that c is a* ***critical number*** *of $f(x)$ (recall: this means c is in the domain of f and either $f'(c) = 0$ or $f'(c)$ DNE). Then:*

- *If f' changes from positive to negative at c, then...* f has a local max at $x=c$.
- *If f' changes from negative to positive at c, then...* f has a local min at $x=c$.
- *If f' does not change sign at c, then...* neither.

1.2.3 Concavity

A function is called **concave upward** on an interval if (equivalently)...

- f' is... increasing on that interval
- f'' is... positive on that interval

A function is called **concave down** on an interval if (equivalently)...

- f' is... decreasing on that interval
- f'' is... negative on that interval.

Remember, $f(x)$ has an **inflection point** at $x = c$ if... f changes concavity at $x=c$

Theorem 1.8 **(Second Derivative Test)** *Suppose $f(x)$ is continuous near c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a... local min @ $x=c$

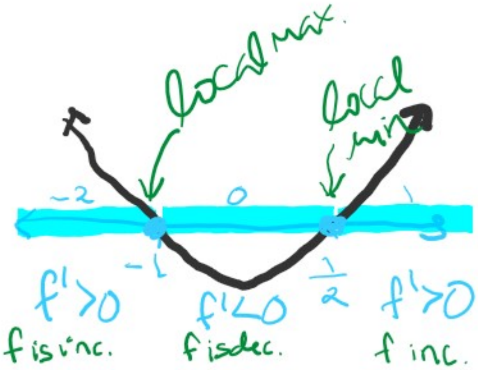
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a... local max @ $x=c$

Example 1.9 $f(x) = 4x^3 + 3x^2 - 6x + 1$ $f' > 0$ $f' < 0$

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity and the inflection points.



a) $f'(x) = 12x^2 + 6x - 6$

$f'(x) = 6(2x^2 + x - 1) = 0$

$6(2x-1)(x+1) = 0$

$x = \frac{1}{2}, \; x = -1$

$f'(-2) = 6(-5)(-1) > 0$

$f'(0) = 6(-1)(1) < 0$

$f'(1) = 6(1)(2) > 0$

$f(\frac{1}{2}) = \frac{1}{2} + \frac{3}{4} - 3 + 1 = \frac{5}{4} - \frac{12}{4} + \frac{4}{4} = -\frac{3}{4}$

a) $\boxed{\begin{array}{l} f \text{ is inc: } (-\infty, -1) \cup (\frac{1}{2}, \infty) \\ f \text{ is dec: } (-1, \frac{1}{2}) \end{array}}$

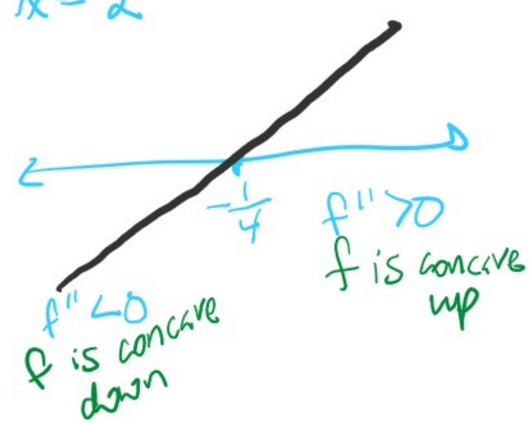
b) f has a local max of $\underbrace{f(-1) = 6}$ at $x = -1$

f has a local min of $\underbrace{f(\frac{1}{2}) = -\frac{3}{4}}$ at $x = \frac{1}{2}$

c) $f''(x) = \underbrace{24x + 6} = 0$

$24x = -6$

$x = \frac{-6}{24} = \frac{-1}{4}$



f concave down: $(-\infty, -\frac{1}{4})$

f concave up: $(-\frac{1}{4}, \infty)$

f has an inflection point at $x = -\frac{1}{4}$

the coords of this pt. $(-\frac{1}{2}, f(-\frac{1}{4}))$

Example 1.10 *From the textbook:*

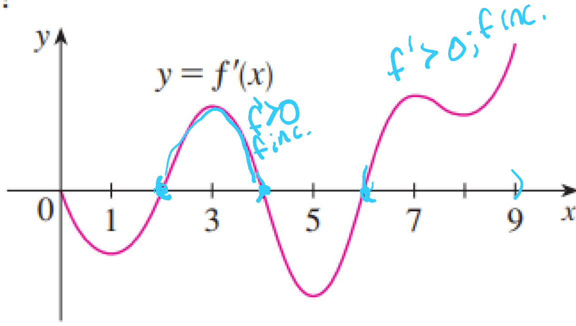
6. The graph of the first derivative f' of a function f is shown.

(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum? Explain.

(c) On what intervals is f concave upward or concave downward? Explain.

(d) What are the x-coordinates of the inflection points of f? Why?



a) f is increasing $\longleftrightarrow$ $f' > 0$

$(2,4) \cup (6,9)$

b) f has a local max $\longleftrightarrow$ f' change from + to −

$x = 4$

f has a local min $\longleftrightarrow$ f' changes from − to +

$x = 2, 6$

c) f concave up $\longleftrightarrow$ $f'' > 0$

$\longleftrightarrow$ f' is increasing

$(1,3), (5,7), (8,9)$

f concave down $\longleftrightarrow$ f' is decreasing (i.e., f' has a neg. slope)

$(0,1), (3,5), (7,8)$

d) f has inflection points at $x = 1, 3, 5, 7, 8$

b/c f changes concavity at these x-values.

$f(-1) = \frac{1}{9}$

Example 1.11 $f(x) = \frac{x^2}{(x-2)^2}$

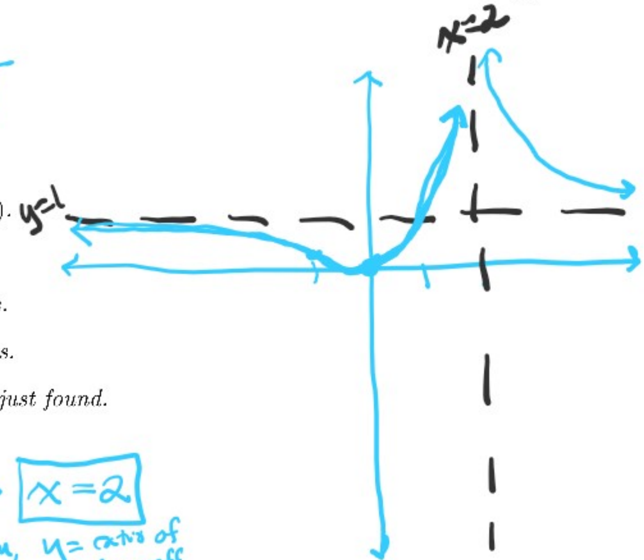
(a) Find the vertical and horizontal asymptotes of $f(x)$.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and local minimum values.

(d) Find the intervals of concavity and inflection points.

(e) Sketch the graph, using all of the information you just found.



a) Vertical Asymptote: set denom $= 0 \longrightarrow \boxed{x=2}$

Horizontal Asymptote: Same deg. in num & denom, $y =$ ratio of leading coeff $\boxed{y=1}$

b) $f'(x) = \frac{\overbrace{(x-2)^2}^{\text{low}} \cdot \overbrace{2x}^{\text{d-hi}} - \overbrace{x^2}^{\text{hi}} \overbrace{(2(x-2))}^{\text{d-low}}}{(x-2)^4}$

$= \frac{(x-2)[(x-2)2x - x^2 \cdot 2]}{(x-2)^4}$

$= \frac{(x-2)2x - 2x^2}{(x-2)^3}$

$= \frac{-4x}{(x-2)^3}$

critical #'s: $f' = 0$ or DNE:

$x = 2, 0$

not in domain of f

$-1 \quad 0 \quad 1 \quad 2$

$f' < 0$ (f dec. $(-\infty, 0)$) $\quad 0 \quad$ $f' > 0$ (f is inc. $(0, 2)$) $\quad 2 \quad$ $f' < 0$ (f dec. $(2, \infty)$)

c) min at $x = 0$ b/c f' goes $-$ to $+$

f goes $\cup$

local min value: $f(0) = 0$

(VA @ $x = 2$ means 2 is not in domain of f)

d) $f'(x) = \frac{-4x}{(x-2)^3}$

$f''(x) = \frac{(x-2)^3(-4) - (-4x) \cdot 3(x-2)^2}{(x-2)^6}$

$f''(x) = \frac{(x-2)^2[(x-2)(-4) + 12x]}{(x-2)^4}$

$f''(x) = \frac{-4x + 8 + 12x}{(x-2)^4}$

$f''(x) = \frac{8x+8}{(x-2)^4}$

$-2 \quad -1 \quad 0 \quad 2 \quad 3$

$f'' < 0$ f is concave down $\quad -1 \quad$ $f'' > 0$ f concave up $\quad 2 \quad$ $f'' > 0$ f concave up

Concave up: $(-1, 2) \cup (2, \infty)$

Concave down: $(-\infty, -1)$

Inflection pt. $(-1, \frac{1}{9})$