04.01 Related Rates

Monday, October 5, 2020 12:31 AM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 4.1: Related Rates

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Today's Goal: Word Problems with Derivatives! Specifically chain rule.

Logistics: We will start this Monday, finish Tuesday and do an activity. On Wednesday we'll take a step back to Section 3.6.

Warm-Up 1.1 *Find y':*

$$\frac{d}{dx}\left(y^2 + 2xy^2\right) = \frac{d}{dx}(1)$$

$$2yy' + 2y^2 + 2x(2yy') = 0$$

$$y'(2y + 4xy) = -2y^2$$

$$y' = \frac{-2y^2}{2y+4xy} = \frac{-y^2}{y+2xy}$$

(A) 1

(B) $\frac{2y}{4xy-2y}$

$\boxed{(C)\ \frac{-y^2}{y+2xy}}$ ←

(D) $\frac{y^2}{2y+xy}$

(E) None of the above.

1.1 Related Rates

Big idea: Use Chain Rule to relate rates of change in real-life examples. Recall Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Remark 1.2 (What variable are you taking the derivative 'with respect to'?) *This is a derivative **with respect to** x (note the dx in the denominator of the notation for the derivative). If we think of f as just being a function of g (and forget that g is a function of x!), we would just write the derivative as*

$\frac{d}{dg} f$ [redacted] $'(g)$ $\qquad \frac{d}{dx}(f(x)) = f'(x)$

It's important to know what variable you are taking the derivative with respect to. In terms of word problems, this would be considering what the units are for your rate of change. $s'(t)$ is velocity, which is a rate of change with respect to time.

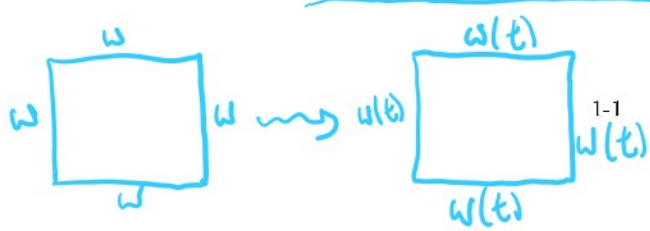
We sometimes use different letters for functions representing real-life quantities. For example, consider area (A) as a function of width (w), and suppose width is a function of time (t):

Area of a square:

- How do we write our area function: $A = w^2$ $\quad A(w) = w^2$
- How do we write the derivative of area with respect to time: $(f(t))^2$

$$A(t) = (w(t))^2$$

$$\boxed{A'(t) = 2w(t) \cdot w'(t)}$$



rate of change of area w/ respect to time

Example 1.3 (Due to Dr. Patrick Newberry) *Oil is spilling out of a tanker in a circular pattern whose radius is increasing at a rate of 3 ft/sec. How fast is the area increasing when the radius is 60ft?*

1) Draw 2 pictures: one capturing movement and one frozen in time.

$r'(t) = 3$ ft/sec

We are being asked to find $A'(t)$ when $r(t) = 60$ ft.

2) rewrite question in terms of derivatives & variables.

$A = \pi r^2$

$A(t) = \pi (r(t))^2$

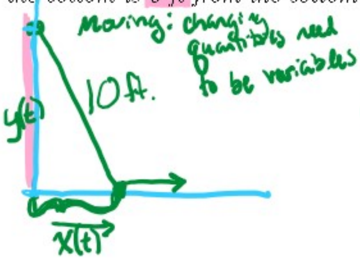
3) Solve:

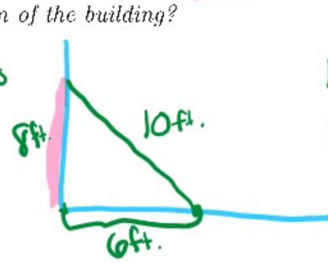
$$A'(t) = 2\pi \overbrace{r(t)}^{60} \cdot \overbrace{r'(t)}^{3}$$

$$A'(t) = 2\pi \cdot 60 \cdot 3$$

$$\boxed{A'(t) = 360\pi \text{ ft}^2/\text{second.}}$$

Example 1.4 (Due to Dr. Patrick Newberry) *A 10 ft ladder is leaning against a building. If the bottom is pulled away from the building at a rate of 2 ft/sec. at what rate is the top of the ladder falling when the bottom is 6 ft from the bottom of the building?*





$10^2 = 6^2 + y^2$

$100 - 36 = y^2$

$64 = y^2$

$8 = y$

☆

$x'(t) = 2$ ft/sec

(x is inc. at this rate!)

Translate question: Want to find $y'(t)$ when $x(t) = 6$ft.

Need: An eq to relate x and y:

$$(x(t))^2 + (y(t))^2 = 10^2$$

$$2x(t)x'(t) + 2y(t)y'(t) = 0$$

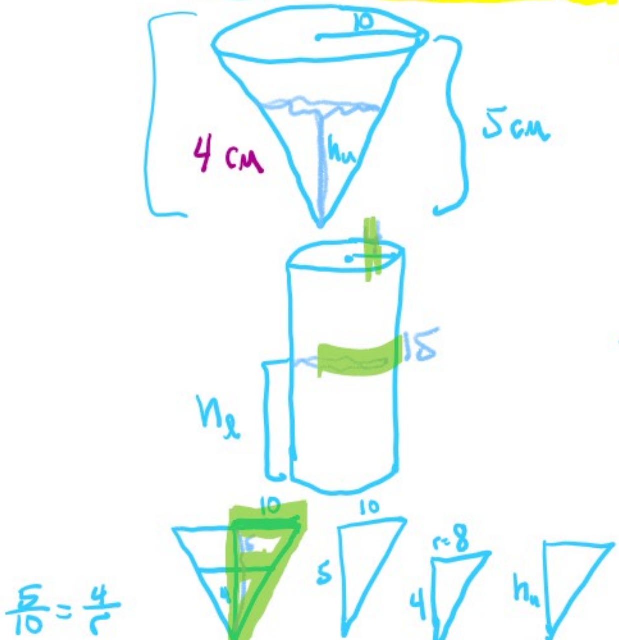
$$2(6)(2) + 2(8)y'(t) = 0$$

$$24 + 16y' = 0 \rightsquigarrow \boxed{y' = -\frac{3}{2} \text{ ft/sec.}}$$

Example 1.5 (Due to Dr. Patrick Newberry) *Water is flowing from a cone with radius 10 cm and height 5 cm into a cylinder with radius 4 cm and height 15 cm. Suppose the water is flowing out of the cone such that the height of the water is decreasing at a rate of 0.25 cm/sec. How fast is the water in the cylinder rising when it is 4 cm deep in the cone?*

$\frac{dh_u}{dt} = -.25$ Want: $\frac{dh_\ell}{dt}$

rewrite r in terms of h_u, b/c we don't have r



cone: $V_u = \frac{1}{3}\pi r^2 h_u \rightsquigarrow V_u(t) = \frac{1}{3}\pi \cdot r(t)^2 \cdot h_u(t)$

$V_u(t) = \frac{1}{3}\pi 4 h_u(t)^3$

cyl: $V_\ell = \pi \underset{4}{\underset{\parallel}{r}}^2 h_\ell \rightsquigarrow V_\ell(t) = \pi \cdot 16 \cdot h_\ell(t)$

Take derivatives: $V_u'(t) = 4\pi (h_u(t))^2 \cdot h_u'(t)$

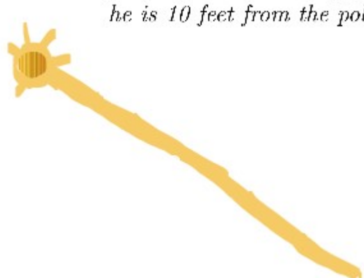
$V_u'(t) = 4\pi \cdot 4^2 \cdot -.25 = -16\pi \text{ cm}^3/\text{sec}$

$= -V_\ell'(t)$!

$V_\ell'(t) = 16\pi h_\ell'(t) = 16\pi$

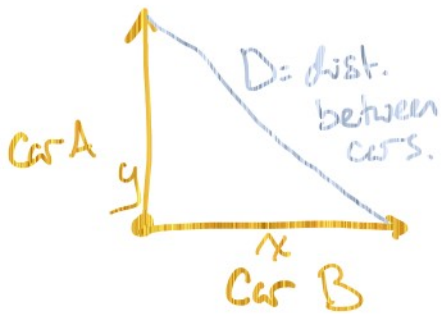
$h_\ell'(t) = 1$ cm/sec.

Example 1.6 (Due to Dr. Patrick Newberry) *A streetlight is mounted at the top of a 15 ft pole. A 6 ft tall man walks away from the pole at a speed of 3 ft/sec. How fast is the tip of the shadow moving when he is 10 feet from the pole?*

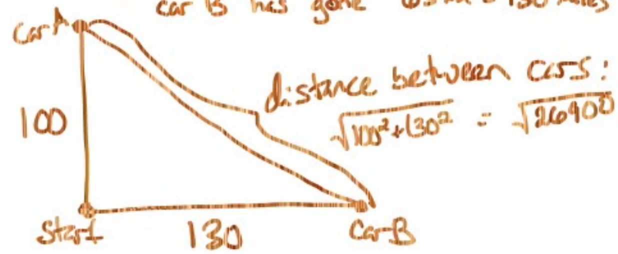


Example 1.7 (Due to Dr. Patrick Newberry) *Two cars start at the same point. Car A travels North at 50 mph and Car B travels East at 65 mph. How fast is the distance between the cars changing after 2 hours?*

after 2h, Car A has gone $50 \times 2 = 100$ miles
car B has gone $65 \times 2 = 130$ miles



Frozen:



$$\frac{dy}{dt} = 50 \qquad \frac{dx}{dt} = 65 \longrightarrow \text{want: } \frac{dD}{dt}.$$

Equation to relate them:

$$D^2 = x^2 + y^2$$

$$D(t)^2 = x(t)^2 + y(t)^2$$

Take $\frac{d}{dt}$

$$2D(t)\underbrace{D'(t)}_{\text{solve for this!}} = 2x(t)x'(t) + 2y(t)y'(t)$$

Plug in known values:

$$2(\sqrt{26900})D'(t) = 2(100)(50) + 2(130)(65)$$

$$D'(t) = \frac{23450}{\sqrt{26900}} \approx \boxed{142.98 \text{ miles per hour}}$$