03.08 Rates of Change

Wednesday, October 14, 2020 10:56 AM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.8: Rates of Change

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Today's Goal: Learn about applications of the derivative to other sciences.

Logistics: We start this on Wednesday and finish it on Friday. There is a check-in on Friday!

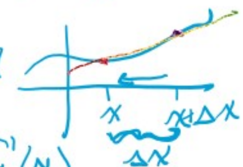
Warm-Up 1.1 *True or False:* $\frac{d}{dx}(\arctan(x^2)) = \frac{2x}{1+x^4}$.

$\frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$ ✓

Big Picture:

$$\text{Average Rate of Change of } f \text{ between time } x \text{ and } x+\Delta x = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\text{Instantaneous rate of change of } f \text{ at time } x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$



1.1 Physics

1.1.1 Motion

$s(t)$ = position, $s'(t) = v(t)$ = velocity ...

Example 1.2 *The position of a particle at time t (seconds) is given $s(t) = \frac{1}{3}t^3 - t$. Draw a diagram to illustrate the motion of the particle. Find the total distance traveled by the particle in the first 8 ~~sections~~ seconds. When is the particle speeding up? When is it slowing down?*

$y = s(t) = \frac{1}{3}t^3 - t$

• x- and y-intercepts

y-int: $s(0) = 0$

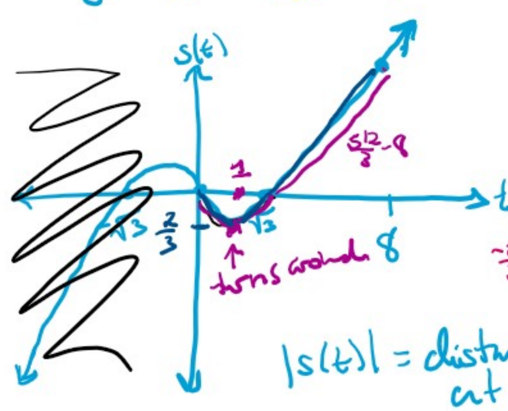
t-int: $0 = \frac{1}{3}t^3 - t$

$= t\left(\frac{1}{3}t^2 - 1\right)$

$t = 0$ | $\frac{1}{3}t^2 - 1 = 0$

$t^2 = 3$

$t = \pm\sqrt{3}$



same sign

$v(t)>0$ and $a(t)>0$ → speeding up

$v(t)<0$ and $a(t)<0$ → speeding up

$v(t)>0$ and $a(t)<0$ → slowing down

$v(t)<0$ and $a(t)>0$ → slowing down.

diff sign

$|s(t)|$ = distance from the start at time t

How far does particle move between $t=\sqrt{3}$ and $t=8$? $s(8) = \frac{1}{3}8^3 - 8$

$= \boxed{\frac{512}{3} - 8}$

Between $t=0$ and $t=\sqrt{3}$, it's important to see when the particle turns around: ie, where $s'(t)=0$.

$s'(t) = t^2 - 1 = 0$

$t = \pm 1$

turns around at $t=1$.

So find $s(1) = \frac{1}{3} - 1 = -\frac{2}{3}$

→ move $\frac{2}{3}$ negative direction

Total movement $= \frac{2}{3} + \frac{2}{3} + \frac{512}{3} - 8 = \boxed{\frac{516}{3} - 8}$

Do sign chart w/ $a(t)$ & $v(t)$:
$v(t) = t^2 - 1$
$a(t) = 2t$

	slowing down		speeding up		slowing down		speeding up
		-1		0		$+1$	
$v(t)$	$+$		$-$		$-$		$+$
$a(t)$	$-$		$-$		$+$		$+$

Speeding up: $(1, \infty)$
Slowing down: $(0, 1)$

1.1.2 Linear Density

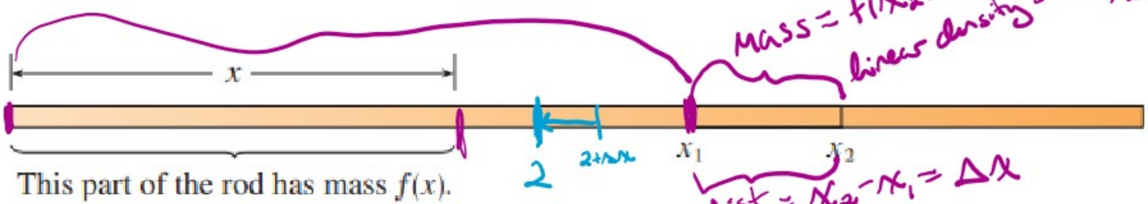
Linear density is defined as mass per unit length, and is usually denotes ρ. Think of a length of wire where the density of the wire changes between end to end.

When the mass is constant, this is easy to calculate: $\rho = \text{mass/length}$. However, if mass is changing we must use a formula:

$$\rho(x) = \lim_{\Delta x \to 0} \frac{m(x + \Delta x) - m(x)}{\Delta x} = m'(x) = f'(x)$$

Note that this is the derivative with respect to length (distance from left end of rod).

mass $= f(x_2) - f(x_1)$
linear density $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$



dist $= x_2 - x_1 = \Delta x$

Example 1.3 *The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $4x^3$ kg. Find the linear density when $x = 2$ meters.*

$m'(x) = 12x^2$ $\quad m'(2) = 12(2)^2 = \boxed{48 \text{ kg/m}}$

$\rho(2) = \lim_{a \to 2} \dfrac{m(2) - m(a)}{2 - a} = m'(2)$

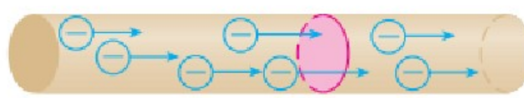
1.1.3 Current is the Derivative of Charge

Average current is given by:

$$\text{average current} = \frac{\Delta Q}{\Delta t}$$

instantaneous current $= \lim_{\Delta t \to 0} \left(\dfrac{\Delta Q}{\Delta t}\right) = Q'(t)$

where ΔQ is the amount of charge that passes through a fixed cross section of a rod over a period of time (Δt time).



Avg. current between $t=0$ and $t=1$ $= \dfrac{\Delta Q}{1-0}$

1.2 Chemistry

1.2.1 Rate of Reaction

The concentration of a reactant A is the number of moles per liter and is denoted by [A]. The concentration varies during a reaction, leading to the definition:

$$\text{Average rate of reaction} = \frac{\Delta[A]}{\Delta t} \rightsquigarrow \text{instantaneous rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[A]}{\Delta t} = [A]'(t)$$

1.2.2 Compressibility

$$\text{Isothermal Compressibility} = \frac{-1}{V}\frac{dV}{dP}, \qquad \boxed{IC = \frac{-1}{V} \cdot V'(P)}$$

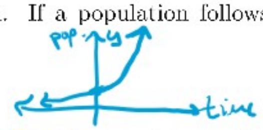
where P is pressure and V is volume.

1.3 Biology

1.3.1 Population Growth

In the past, you have probably encountered **exponential growth**. If a population follows exponential growth, then the population at time t can be expressed:

$$\boxed{P(t) = Ae^{kt}}$$



where A is the population at time $t = 0$, and k is a growth constant depending on the population.

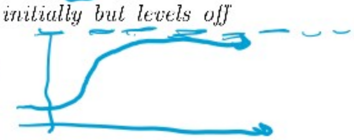
Differentiate to get the population growth rate:

$$P'(t) = kAe^{kt} \to P'(t) = kP(t)$$

Exp. growth is not the only kind...

Example 1.4 *The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function*

$$a(1+be^{-.7t})^{-1} \longrightarrow n = f(t) = \frac{a}{1 + be^{-0.7t}} \leftarrow \text{not exp.}$$



$$n = \frac{140}{1+6e^{-.7t}}$$

$\to 0$ as $t \to \infty$, $n \to 140$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run.

at $t=0$: $f(0)=20$, $f'(0)=12$

$$f(0) = \frac{a}{1+be^{-.7 \cdot 0}} = \frac{a}{1+b}$$

① $\boxed{20 = \frac{a}{1+b}} \to a = 20(1+b)$ $\boxed{a = 140}$

$$f'(t) = -1 \cdot a(1+be^{-.7t})^{-2} \cdot (-.7be^{-.7t})$$

$$f'(t) = .7abe^{-.7t}(1+be^{-.7t})^{-2}$$

$f'(0) = 12$: $f'(0) =$ ② $\boxed{\frac{.7ab}{(1+b)^2} = 12}$

$$\frac{.7(20)\cancel{(1+b)}b}{(1+b)^{\cancel{2}}} = 12$$

$$14b = 12(1+b)$$

$$2b = 12 \quad \boxed{b = 6}$$

1.4 Economics

1.4.1 Marginal Cost

The concept of average cost per unit is an important concept in production. If $C(x)$ is the total cost a company incurs producing x units of their products, then we can find the average rate of change of cost:

$$\boxed{\frac{C(x+\Delta x)-C(x)}{\Delta x}}$$

Taking the limit as $\Delta x \to 0$, we obtain an instantaneous rate of change of cost, which is referred to as the **marginal cost**:

$$\text{marginal cost} = \frac{dC}{dx}$$

Example 1.5 *The cost function for production of a commodity is $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$. Find and interpret $C'(100)$.*
Compare $C'(100)$ with the cost of producing the 101st item.

$C'(x) = 25 - 0.18x + 0.0012x^2$ 10,000

$C'(100) = 25 - 18 + 120 = \boxed{127}$

$C(100) = 2{,}339$

$C(101) = 2{,}358$

$\frac{C(101)-C(100)}{101-100} = \boxed{19}$

$C(99) \to C(100)$