

# 03.08 Rates of Change

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Lecture: Section 3.8: Rates of Change

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**Today's Goal: Learn about applications of the derivative to other sciences.**

Logistics: We start this on Wednesday and finish it on Friday. There is a check-in on Friday!

★ Warm-Up 1.1 True or False:  $\frac{d}{dx}(\arctan(x^2)) = \frac{2x}{1+x^2}$ .   
 $\frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$  ✓

Big Picture:

Average Rate of Change of  $f$  between time  $x$  and  $x + \Delta x = \frac{f(x + \Delta x) - f(x)}{\Delta x}$    
 Instantaneous rate of change of  $f$  at time  $x = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

1.1 Physics

1.1.1 Motion  $s(t)$  = position,  $s'(t) = v(t)$  = velocity ...

Example 1.2 The position of a particle at time  $t$  (seconds) is given  $s(t) = \frac{1}{3}t^3 - t$ . Draw a diagram to illustrate the motion of the particle. Find the total distance traveled by the particle in the first 8 seconds. When is the particle speeding up? When is it slowing down?

$y = s(t) = \frac{1}{3}t^3 - t$  • x- and y- intercepts

$y\text{-int} = s(0) = 0$   
 $t\text{-int} = 0 = \frac{1}{3}t^3 - t$   
 $= t(\frac{1}{3}t^2 - 1)$   
 $t=0 \mid \frac{1}{3}t^2 - 1 = 0$   
 $t^2 = 3$   
 $t = \pm\sqrt{3}$

same sign  $\rightarrow$  speeding up  
 $v(t) > 0$  and  $a(t) > 0$   
 $\rightarrow$  speeding up  
 $v(t) < 0$  and  $a(t) < 0$   
 $\rightarrow$  speeding up  
 $v(t) > 0$  and  $a(t) < 0$   
 $\rightarrow$  slowing down  
 $v(t) < 0$  and  $a(t) > 0$   
 $\rightarrow$  slowing down.

diff sign

$|s(t)|$  = distance from the start at time  $t$

How far does particle move between  $t = \sqrt{3}$  and  $t = 8$ ?  $s(8) = \frac{1}{3}8^3 - 8 = \frac{512}{3} - 8$

Between  $t = 0$  and  $t = \sqrt{3}$ , it's important to see when the particle turns around: i.e. where  $s'(t) = 0$ .

$s'(t) = t^2 - 1 = 0$  turns around at  $t = 1$ .   
 $t = \pm 1$    
 so find  $s(1) = \frac{1}{3} - 1 = -\frac{2}{3}$    
 $\rightarrow$  move  $\frac{2}{3}$  negative direction   
 Total movement =  $\frac{2}{3} + \frac{2}{3} + \frac{512}{3} - 8 = \frac{516}{3} - 8$

1-2

Do sign chart  
w/  $a(t)$  &  $v(t)$ :  
 $v(t) = t^2 - 1$   
 $a(t) = 2t$

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slowing down   speeding up   slowing down   speeding up

$v(t)$   
 $a(t)$

+	-	-	0	-	+	+
-	-	-	0	+	+	+

Speeding up:  
(1, ∞)  
Slowing down  
(0, 1)

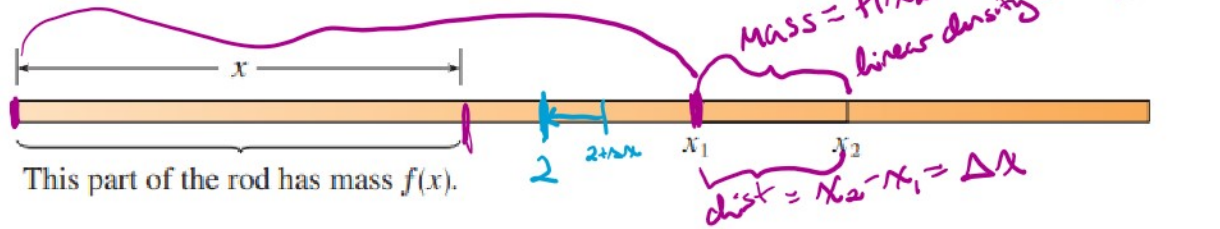
### 1.1.2 Linear Density

**Linear density** is defined as mass per unit length, and is usually denoted  $\rho$ . Think of a length of wire where the density of the wire changes between end to end.

When the mass is constant, this is easy to calculate:  $\rho = \text{mass}/\text{length}$ . However, if mass is changing we must use a formula:

$$\rho(x) = \lim_{\Delta x \rightarrow 0} \frac{m(x + \Delta x) - m(x)}{\Delta x} = m'(x) = f'(x)$$

Note that this is the derivative with respect to length (distance from left end of rod).



**Example 1.3** The mass of the part of a metal rod that lies between its left end and a point  $x$  meters to the right is  $4x^3$  kg. Find the linear density when  $x = 2$  meters.

$m'(x) = 12x^2$     $m'(2) = 12(2)^2 = 48 \text{ kg/m}$

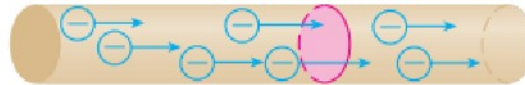
$$\rho(2) = \lim_{a \rightarrow 2} \frac{m(2) - m(a)}{2 - a} = m'(2)$$

### 1.1.3 Current is the Derivative of Charge

Average current is given by:

average current =  $\frac{\Delta Q}{\Delta t}$    instantaneous current =  $\lim_{\Delta t \rightarrow 0} \left( \frac{\Delta Q}{\Delta t} \right) = Q'(t)$

where  $\Delta Q$  is the amount of charge that passes through a fixed cross section of a rod over a period of time ( $\Delta t$  time).



Avg. current between  $t=0$  and  $t=1$  =  $\frac{\Delta Q}{1-0}$

## 1.2 Chemistry

### 1.2.1 Rate of Reaction

The **concentration** of a reactant  $A$  is the number of moles per liter and is denoted by  $[A]$ . The concentration varies during a reaction, leading to the definition:

$$\text{Average rate of reaction} = \frac{\Delta[A]}{\Delta t} \rightarrow \text{instantaneous rate of reaction} = \lim_{\Delta t \rightarrow 0} \frac{\Delta[A]}{\Delta t} = [A]'(t)$$

### 1.2.2 Compressibility

$$\text{Isothermal Compressibility} = -\frac{1}{V} \frac{dV}{dP},$$

where  $P$  is pressure and  $V$  is volume.

## 1.3 Biology

### 1.3.1 Population Growth

In the past, you have probably encountered **exponential growth**. If a population follows exponential growth, then the population at time  $t$  can be expressed:

$$P(t) = Ae^{kt}$$

where  $A$  is the population at time  $t = 0$ , and  $k$  is a growth constant depending on the population.

Differentiate to get the population growth rate:

$$P'(t) =$$

**Example 1.4** The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where  $t$  is measured in hours. At time  $t = 0$  the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of  $a$  and  $b$ . According to this model, what happens to the yeast population in the long run.

## 1.4 Economics

### 1.4.1 Marginal Cost

The concept of average cost per unit is an important concept in production. If  $C(x)$  is the total cost a company incurs producing  $x$  units of their products, then we can find the average rate of change of cost:

$$\frac{C(x + \Delta x) - C(x)}{\Delta x}$$

Taking the limit as  $\Delta x \rightarrow 0$ , we obtain an instantaneous rate of change of cost, which is referred to as the **marginal cost**:

$$\text{marginal cost} = \frac{dC}{dx}$$

**Example 1.5** *The cost function for production of a commodity is  $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$ . Find and interpret  $C'(100)$ . Compare  $C'(100)$  with the cost of producing the 101st item.*