03.07 Log Derivatives

Sunday, October 11, 2020 9:52 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.7: Derivatives of Logarithmic Functions

Lecturer: Sarah Arpin

Today's Goal: Learn about the derivatives of log functions.

Logistics: We start and finish this on Monday. There is an evening quiz Tuesday this week - SET AN ALARM! - Projects 6, 7 (M.) – 3.3, 3.4, 3.5, 3.6, 4.1

Warm-Up 1.1 *Find $f'(1/2)$ for $f(x) = (\arcsin(x))^2$.*

(A) $\frac{2\pi}{3\sqrt{3}}$

(B) $\frac{-2\pi}{3}$

(C) 0

(D) 6π

(E) None of the above.

$$f'(x) = 2(\arcsin(x)) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$f'(1/2) = 2\left(\frac{\pi}{6}\right) \cdot \frac{1}{\sqrt{3/4}} \rightarrow \frac{2}{\sqrt{3}}$$

$$= \frac{4\pi}{6\sqrt{3}} = \boxed{\frac{2\pi}{3\sqrt{3}}}$$

1.1 Derivatives of Logarithmic Functions

1.1.1 The Natural Log - Revisited

Recall the natural log function

$$y = \ln(x)$$

This is the function whose output is the exponent you raise e to in order to get the value x. For example, $\ln(e^4) = 4$.

This makes $\ln(x)$ the inverse of the exponential function e^x. Inverse functions have the special property $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. Using this property, we can now prove the derivative of $\ln(x)$:

ln f

$$e^{\ln(x)} = x$$

$$\frac{d}{dx}\left(e^{\ln(x)}\right) = \frac{d}{dx}(x)$$

$$e^{\ln(x)} \cdot \left[\frac{d}{dx}\ln(x)\right] = 1$$

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}$$

$= x$ b/c inverses.

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

1.1.2 Logarithms of Arbitrary Base

and b^x, $\log_b(x)$ are inverses

Remembering the fact that $\frac{d}{dx}b^x = \ln(b)b^x$, we can use the same algorithm as above to prove the derivative of the function $f(x) = \log_b(x)$, for any valid base b:

$$b^{\log_b(x)} = x$$

$$\frac{d}{dx}\left(b^{\log_b(x)}\right) = \frac{d}{dx}(x)$$

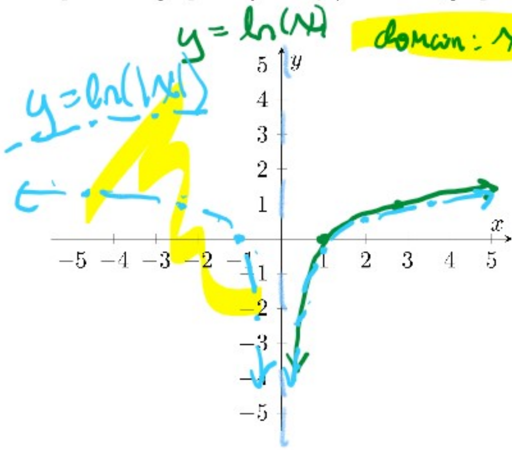
$$\ln(b)\cdot b^{\log_b(x)} \cdot \left[\frac{d}{dx}\log_b(x)\right] = 1$$

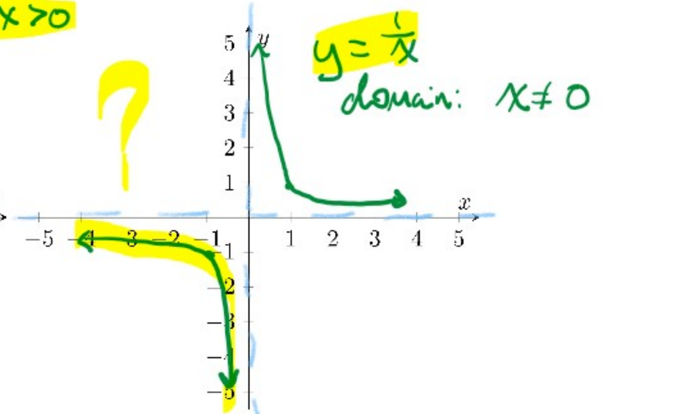
$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{\ln(b)\cdot b^{\log_b(x)}}$$

$$\boxed{\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\ln(b)}}$$

1.2 The Catch : Be careful of domain

Let's compare the graph of $y = \ln(x)$ with the graph of its derivative $y' = \frac{1}{x}$.





This leads us to a new rule:

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ for } x \neq 0 \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x} \text{ when } x > 0$$

Keep this in mind for when we start talking about anti-derivatives in Chapter 5.

1.3 Quick Review of Properties of Log Functions

Stated for $\log_a(x)$, but valid for every base log function:

1. $\log_a(x^c) = c\log_a(x)$
2. $\log_a(xy) = \log_a(x) + \log_a(y)$
3. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

1.4 Logarithmic Differentiation

Sometimes, taking the log of a function first will actually simplify the computation of a derivative. For example, consider:

$$y = \frac{x^3\sin(2x)}{e^{x^2}\sqrt[3]{x+1}}$$

Finding y' would be an annoying (but not impossible) application of Quotient Rule. However, applying the ln function to both sides of the equation would allow us to use properties of logarithms to expand into several terms and may actually make the computation simpler in the end! We will need chain rule, but that is no problem for us.

$$\ln(y) = \ln\left(\frac{x^3\sin(2x)}{e^{x^2}\sqrt[3]{x+1}}\right)$$

① apply ln to both sides
② apply $\frac{d}{dx}$ to both sides
③ solve for y'

Let's separate it out and then apply $\frac{d}{dx}$ to both sides to find y':

$$\ln(y) = \ln(x^3\sin(2x)) - \ln(e^{x^2}\cdot(x+1)^{1/3})$$

$$\ln(y) = 3\ln(x) + \ln(\sin(2x)) - [x^2\ln(e) + \tfrac{1}{3}\ln(x+1)]$$

$$\ln(y) = 3\ln(x) + \ln(\sin(2x)) - x^2 - \tfrac{1}{3}\ln(x+1)$$

Apply $\frac{d}{dx}$:

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(3\ln(x) + \ln(\sin(2x)) - x^2 - \frac{1}{3}\ln(x+1)\right)$$

$$\frac{1}{y}\cdot y' = 3\cdot\frac{1}{x} + \frac{1}{\sin(2x)}\cdot\cos(2x)\cdot 2 - 2x - \frac{1}{3}\cdot\frac{1}{x+1}$$

$$y' = y\left(\frac{3}{x} + \frac{2\cos(2x)}{\sin(2x)} - 2x - \frac{1}{3(x+1)}\right)$$

$$y' = \frac{x^3\sin(2x)}{e^{x^2}\sqrt[3]{x+1}}\cdot\left(\frac{3}{x} + \frac{2\cos(2x)}{\sin(2x)} - 2x - \frac{1}{3(x+1)}\right)$$

1.5 Examples

we have no rule for this!

Example 1.2 *Find y' for $y = x^x$ using logarithmic differentiation.*

① Apply ln: $\ln(y) = \ln(x^x)$

$\ln(y) = x \cdot \ln(x)$

② Apply $\frac{d}{dx}$: $\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(x))$

$\frac{1}{y} \cdot y' = \ln(x) + x \cdot \frac{1}{x}$

$\frac{1}{y} y' = \ln(x) + 1$

$y' = y(\ln(x) + 1)$

$\boxed{y' = x^x(\ln(x) + 1)}$

* what about using other log's?

$\log(y) = \log(x^x)$ (base 10)

$\log(y) = x \log(x)$

$\frac{d}{dx}$: $\frac{1}{\ln(10) y} \cdot y' = \ldots$

Example 1.3 *Find $\frac{dy}{dx}$ for $y = \sin^2(2^x)$.*

$y = (\sin(2^x))^2$

$y' = 2\sin(2^x) \cdot \cos(2^x) \cdot \ln(2) 2^x$

$\boxed{\frac{dy}{dx} = 2\ln(2) 2^x \cdot \sin(2^x)\cos(2^x)}$

$f'(x) = 6x^2 + 17$ $f'(2) = 6 \cdot 4 + 17$

Example 1.4 *If $f(x) = 2x^3 + 17x$, find $(f^{-1})'(50)$ without finding f^{-1}. Use the fact that $f(2) = 50$. Hint: Use the fact that $f(f^{-1}(x)) = x$ to help get a general expression for $(f^{-1})'(x)$.* $= \frac{d}{dx} f^{-1}(x)$

$f(f^{-1}(x)) = x$

Apply $\frac{d}{dx}$:

$\frac{d}{dx}(f(f^{-1}(x))) = \frac{d}{dx}(x)$

$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1$

$\boxed{\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}}$

$(f^{-1})'(50) = \frac{1}{f'(f^{-1}(50))}$

$= \frac{1}{f'(2)}$

$= \frac{1}{24+17} = \boxed{\frac{1}{41}}$

Example 1.5 *Find the equation of the tangent line to $y = (\ln(x^2 + 1))^3$ at the point where $x = 1$*

$x = 1$, $y = \ln(1+1)^3 = \ln(2)^3$

$y' = 3\ln(x^2+1)^2 \cdot \frac{1}{x^2+1} \cdot 2x$

y' @ $x = 1$: $3\ln(2)^2 \cdot \frac{1}{2} \cdot 2 = 3\ln(2)^2$

$\boxed{y - \ln(2)^3 = 3\ln(2)^2(x-1)}$