03.05 Implicit Differentiation

Thursday, October 1, 2020 11:53 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.5: Implicit Differentiation

Lecturer: Sarah Arpin

Today's Goal: Find derivatives when a function isn't given as just $f(x) = \cdots$

Logistics: We will start and finish this section today, Friday. We also have a check-in in the last five minutes.

Warm-Up 1.1 *Find $f'(\pi)$ for $f(x) = (\sin^2(x) - x) \cdot (\cos(x) + 2)$.*

(A) 0

(B) 1

(C) −1

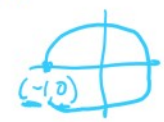
(D) 1 − π

(E) None of the above

$f'(x) = (2\sin x \cos x - 1)(\cos x + 2) + (\sin^2 x - x)(-\sin(x))$

$f'(\pi) = (2 \cdot 0 \cdot (-1) - 1)(-1 + 2) + \cancel{(0^2 - \pi)(-0)}$

$= -1 \cdot 1 = \boxed{-1}$

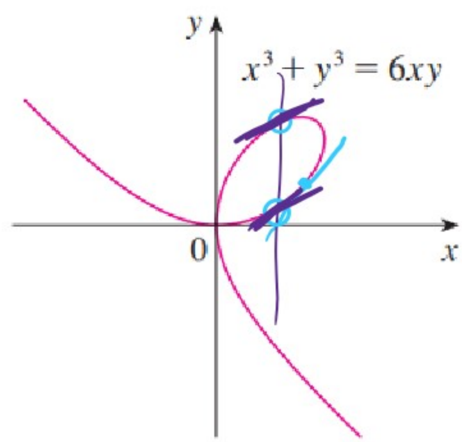


$(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

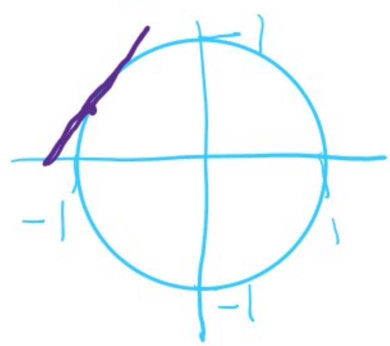
$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

1.1 Implicit Differentiation

So far, we have been given an **explicit** function $f(x)$ or y wjere one side of the equation is written totally in terms of x. For example, we see things like $f(x) = x^2 - 2$, not ones with x's and y's on both sides of the equation, like $x^3 + y^3 = 6xy$, for example.



$x^2 + y^2 = 1$



However, functions like $x^3 + y^3 = 6xy$ are important (this one is called *the Folium of Descartes*), and we would still like to be able to find slopes of tangent lines!

Key technique: Think of *applying the derivative* to both sides of an equation. For example, if we start with a function we know:

$$y = x^2 + 1$$

we can apply the derivative to both sides of the equation as such:

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2+1)$$

and simplify using our knowledge of derivatives to get an expression for $\frac{dy}{dx}$:

$$\left(y' =\right) \frac{dy}{dx} = 2x$$

Apply this same technique to an implicit equation, just be sure to treat y as a function of x, instead of just another variable. Let's start with the folium of Descartes:

$$x^3 + y^3 - 6xy$$

* y is a function of x: $f(x)$, $y(x)$

Begin by *applying the derivative* to both sides of the equation, and we'll see how to interpret this as we go:

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$ ← product rule

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6x)\cdot(y) + 6x\cdot\frac{d}{dx}(y)$$

$$3x^2 + 3y^2 y' = 6y + 6xy'$$

$$3y^2 \frac{dy}{dx}$$

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$3y^2y' - 6xy' = 6y - 3x^2$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

This derivative depends on both x and y!

Example 1.2 (An example we can visualize!) *(a) Find* $\frac{dy}{dx}$ *for* $x^2 + y^2 = 25$.

(b) Use the expression in part (a) to get the equation of a tangent line to the graph at the point $(-3, 4)$.

need slope and point (we have the point) ✓

a) $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$

$2x + 2yy' = 0$

$y' = \frac{-2x}{2y}$

$y' = \frac{-x}{y}$

b) slope $= y'$ @ $(-3,4)$:

$y' = \frac{-(-3)}{4} = \frac{3}{4}$

$$\boxed{y-4 = \frac{3}{4}(x+3)}$$

Example 1.3 *Find* y' *if* $\underbrace{\sin(x-y)}_{\text{chain rule}} = \underbrace{\frac{\cos(x)}{x^2+y^2+1}}_{\text{quotient rule, chain rule for "}y^2\text{"}}$

$\frac{d}{dx}(\sin(x-y)) = \frac{d}{dx}\left(\frac{\cos x}{x^2+y^2+1}\right)$

$\cos(x-y)(1-y') = \frac{(x^2+y^2+1)(-\sin x) - \cos(x)\cdot(2x+2yy')}{(x^2+y^2+1)^2}$

Need to get y''s to same side of equation:

$(x^2+y^2+1)^2\cos(x-y)(1-y') = (x^2+y^2+1)(-\sin x) - \cos x\cdot(2x+2yy')$

distribute:

$(x^2+y^2+1)^2\cos(x-y) - y'(x^2+y^2+1)^2\cos(x-y) = (x^2+y^2+1)(-\sin x) - 2x\cos x - 2yy'\cos x$

move y' stuff to left, other terms to right side:

$2yy'\cos x - y'(x^2+y^2+1)^2\cos(x-y) = (x^2+y^2+1)(-\sin x) - (x^2+y^2+1)^2\cos(x-y)$

Factor + solve for y':

$y'(2y\cos x - (x^2+y^2+1)^2\cos(x-y)) = \text{RHS}$

$\rightarrow y' = \frac{(x^2+y^2+1)(-\sin x) - (x^2+y^2+1)^2\cos(x-y)}{(2y\cos x - (x^2+y^2+1)^2\cos(x-y))}$

phew!

$$x^{1/4} - y^{1/4} = 12$$

Example 1.4 *Find* y'' *if* $\sqrt[4]{x} - \sqrt[4]{y} = 12$.

Apply $\frac{d}{dx}$ to both sides

$$\frac{1}{4}x^{-3/4} - \frac{1}{4}y^{-3/4} \cdot y' = 0$$

Solve for y':

$$y' = \frac{x^{-3/4}}{y^{-3/4}}$$

$$y' = \frac{\sqrt[4]{y^3}}{\sqrt[4]{x^3}}$$

apply $\frac{d}{dx}$ to both sides:

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{y^{3/4}}{x^{3/4}}\right)$$

$$y'' = \frac{x^{3/4} \cdot \left(\frac{3}{4}y^{-1/4} \cdot y'\right) - y^{3/4} \cdot \frac{3}{4}x^{-1/4}}{(x^{3/4})^2}$$

We know this!

$$y'' = \frac{\frac{3x^{3/4}}{4y^{1/4}}\, y' - \frac{3y^{3/4}}{4x^{1/4}}}{x^{6/4}}$$

cancel!

$$y'' = \frac{\frac{\cancel{3\sqrt[4]{x^3}}}{4\cancel{\sqrt[4]{y}}} \cdot \frac{\sqrt[4]{y^3}^{\,2}}{\cancel{\sqrt[4]{x^3}}} - \frac{3\sqrt[4]{y^3}}{4\sqrt[4]{x}}}{x^{3/2}} \Rightarrow \boxed{y'' = \frac{\frac{3}{4}\sqrt{y} - \frac{3\sqrt[4]{y^3}}{4\sqrt[4]{x}}}{x\sqrt{x}}}$$

Example 1.5 *Find equations of both the tangent lines to the ellipse* $x^2 + 4y^2 = 36$ *that pass through the point* ~~(12, 3)~~. oops! $x = 4$

Apply $\frac{d}{dx}$ to both sides:

$$2x + 8yy' = 0$$

$$y' = \frac{-2x}{8y}$$

$$\boxed{y' = \frac{-x}{4y}}$$

$x = 4$ corresponds to two points:

$$4^2 + 4y^2 = 36$$

$$16 + 4y^2 = 36$$

$$4y^2 = 20$$

$$y^2 = 5$$

$$y = \pm\sqrt{5}$$

First point: $(4, \sqrt{5})$:

$$y' = \frac{-4}{4\sqrt{5}} = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$$

Tangent line:

$$\boxed{y - \sqrt{5} = \frac{-\sqrt{5}}{5}(x - 4)}$$

Second point: $(4, -\sqrt{5})$:

$$y' = \frac{-4}{-4\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Tangent line:

$$\boxed{y + \sqrt{5} = \frac{\sqrt{5}}{5}(x - 4)}$$