03.04 The Chain Rule

Wednesday, September 30, 2020 12:37 PM

9-29

Tuesday, September 29, 2020 11:26 AM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.4: The Chain Rule

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Today's Goal: Learn how to deal with derivatives of functions like $f(g(x))$

Logistics: We will start this on Tuesday and finish it on Wednesday. Wednesday we'll also have an activity. Thursday's activity will be mooooore chain rule!
Evening quiz tonight! Set an alarm on your phone!!!

Warm-Up 1.1 *Find $f'(\pi/3)$ for $f(x) = x\tan(x)$.*

$f' = 1 \cdot \tan x + x \sec^2 x.$

$f'\left(\frac{\pi}{3}\right) = \boxed{\frac{\pi}{\sqrt{3}} + \frac{\pi}{3} \cdot 4}$

1.1 Composite Functions

In Precalculus, we learned about the definition of $(f \circ g)(x)$. Let's recall some of this by recognizing some algebraic expressions as a composition of two functions $f(g(x))$:

(1) $\sqrt{2x-1}$ $\boxed{f(x) = \sqrt{x} \quad g = 2x-1}$

(2) $\boxed{\frac{1}{2^x+1}}$ $f(x) = \frac{1}{x}$ $g(x) = 2^x + 1.$

(3) $\sin(2x^2+1)$ $f(x) = \sin x.$ $\boxed{g(x) = 2x^2+1}$

1.2 The Chain Rule

We have not learned how to take the derivative of every function quite yet. What about functions of the form:

$$\boxed{f(x) = \sqrt{2x-1}?}$$

We know how to do the derivative of $\sqrt{x}$ and $2x - 1$ separately, but with them composed like that things will be different.

$(\sqrt{x})' = (x^{\frac{1}{2}})'$
$= \frac{1}{2}x^{-\frac{1}{2}}$
$= \frac{1}{2\sqrt{x}}.$

The **Chain Rule** will help us deal with derivatives of **composite functions**.

Definition 1.2 (The Chain Rule) *If a $g(x)$ is a differentiable function at x and $f(x)$ is a differentiable function at $g(x)$, then we can define the derivative of the composite function $f \circ g$ to be:*

$$(f \circ g)'(x) = \boxed{f'(g(x))} \cdot g'(x)$$

Equivalently, in Liebniz notation we can set $y = f(u)$ and $u = g(x)$ to write the chain rule as:

$$\frac{dy}{dx} = \boxed{\frac{df}{du}}\boxed{\frac{du}{dx}}$$

The key to using the chain rule is **recognizing which function is the "inner" function and which function is the "outer" function in a composition.**

$F = \sqrt{blah}$

$F = \sqrt{blah}$

$F' = \underbrace{\frac{1}{2}(blah)^{-\frac{1}{2}}} \cdot \underbrace{blah'}.$

Example 1.3 *Find the derivative of* $F(x) = \sqrt{2x-1}$ *using the chain rule.*

$F(x) = f(g(x)) \qquad f(x) = \sqrt{x} \qquad g(x) = 2x-1.$

$F'(x) = f'(g(x)) \cdot g'(x).$

$= \frac{1}{2}(2x-1)^{-\frac{1}{2}} \cdot 2. = \frac{1}{\sqrt{2x-1}}.$

$f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$

$f'(g(x)) = \frac{1}{2}(2x-1)^{-\frac{1}{2}}$

$g'(x) = 2.$

Example 1.4 *Find the equation of the tangent line to the curve* $y = \sin(x^2-1)$ *at* $x = 0$.

$y' = \cos(x^2-1)(x^2-1)'$

$\cos(x^2-1) \cdot 2x.$

$y'(0) = \cos(-1) \cdot 2 \cdot 0 = 0$

Example 1.5 *Find the derivative of the function* $F(x) = \frac{x^2+1}{x^2-1}$. 3

blah

$F' = 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \left(\frac{x^2+1}{x^2-1}\right)' = \boxed{3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2}}$

Example 1.6 *Find the equation of the derivative of* $F(x) = \frac{2}{\sqrt{x^2+x+1}}$. $= 2(blab)^{-\frac{1}{2}}$

$f_1 = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}} \qquad g_1 = x^2+x+1.$

$f_1' = 2(-\tfrac{1}{2})x^{-3/2} \qquad g_1' = 2x+1.$

$2(-\tfrac{1}{2})(x^2+x+1)^{-3/2} \quad (x^2+x+1)'$

$2(-\tfrac{1}{2})(x^2+x+1)^{-3/2}(2x+1).$

$F'(x) = f_1'(g_1(x))\, g_1'(x)$

$2(-\tfrac{1}{2})(x^2+x+1)^{-3/2} \cdot (2x+1)$

Example 1.7 *Use the following table of values to find the following derivatives:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	0	-1	-1
0	0	2	1	1
1	-1	1	0	2
2	0	1	-1	2

$(f\circ g)'(x) = f'(g(x))\cdot g'(x)$

1. $(f\circ g)'(2) = f'(g(2))\cdot g'(2) = f'(-1)\cdot 2 = 0$
2. $(g\circ g)'(1)$
3. $(g\circ x^2)'(-1)$

Example 1.8 *Determine the intervals where the function* $f(x) = (x^3+2x^2-3x)^4$ *is increasing/decreasing.*

$f'(x) = 4(x^3+2x^2-3x)^3\cdot(3x^2+4x-3)$

$f'(x) > 0 \rightarrow f(x)$ increasing

$f'(x) < 0 \rightarrow f(x)$ decreasing

> use desmos to see.

Example 1.9 *Find the derivative of the function* $h(t) = \sin(\sqrt{t^3-1})$. $(t^3-1)^{1/2}$

$h'(t) = \cos(\sqrt{t^3-1})\cdot\frac{1}{2}(t^3-1)^{-1/2}\cdot 3t^2$

$$\boxed{h'(t) = \frac{3t^2\cos(\sqrt{t^3-1})}{2\sqrt{t^3-1}}}$$

Example 1.10 *Find the equation of the tangent line to the curve* $f(x) = \sin(\sin(x^2+\pi))$ *at* $x = 0$.

slope, point

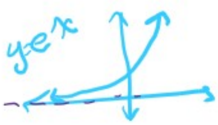
$f'(x) = \cos(\sin(x^2+\pi))\cdot\cos(x^2+\pi)\cdot 2x$

$f'(0) = 0$

$f(0) = \sin(\sin(0^2+\pi))$
$= 0$

"$y = \#$" ← line of slope 0.

$$\boxed{y=0}$$



$y = f(g(x))$ w/ $f(x) = e^x$, $g(x) = rx$ $\quad y' = f'(g(x)) \cdot g'(x)$

Example 1.11 *For what value(s) of r do the derivatives of the function $y = e^{rx}$ satisfy the equation:*

$$y'' - 4y' + y = 0$$

$y' = e^{rx} \cdot r = re^{rx}$

$y'' = r \cdot re^{rx} = r^2 e^{rx}$

$\left(y^{(40)} = r^{40} e^{rx} \text{ pattern!}\right)$

$r^2 e^{rx} - 4re^{rx} + e^{rx} = 0.$

$e^{rx}(r^2 - 4r + 1) = 0.$

does not contribute a 0

solve by quadratic formula:

$$r = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = \boxed{2 \pm \sqrt{3}}$$

Example 1.12 *Write a function that would use the chain rule, product rule, and quotient rule to differentiate. Then differentiate it.*

Be creative!

Example 1.13 *Find the derivative of the function $y = \sqrt{\sqrt{3x+1} - x}$.* $= \left((3x+1)^{1/2} - x\right)^{1/2}$

lots of composition! peel off the layers:

$$y' = \frac{1}{2}\Big(\underbrace{(3x+1)^{1/2} - x}_{\text{inner function}}\Big)^{-1/2} \cdot \left(\frac{1}{2}(\underbrace{3x+1}_{\text{inner function}})^{-1/2} \cdot 3 - 1\right)$$