03.02 The Product and Quotient Rules

Sunday, September 20, 2020 4:00 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.2 The Product and Quotient Rules

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Today's Goal: Deal with derivatives of functions $\boxed{f(x)g(x)}$, $\frac{f(x)}{g(x)}$

Logistics: We will start this on a Wednesday and finish on a Friday. Friday will be a check-in!

Warm-Up 1.1 *True or $\boxed{\text{False}}$.* $\frac{d}{dx}e^x = xe^{x-1}$ variable is in the exponent!

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}b^x = \ln(b)\cdot b^x$$

$$\frac{d}{dx}x^3 = 3x^2$$
variable is base

1.1 Product Rule

We want to be able to take the derivative of products of functions that we can't distribute, for example:

$$h(x) = e^x\sqrt{5x} = e^x \cdot \sqrt{5}\cdot\sqrt{x}$$
don't have a rule yet!

is the product of the two functions:
Our previous rules do not apply! We need a new rule! Let's go back to the definition of derivative and see if we get anywhere:

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

A cool trick that is frequently helpful in math is to add and subtract a term, and see if that helps:

$$\frac{A+B}{C} = \frac{A}{C}+\frac{B}{C}$$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h)-f(x)]}{h}$$

$$= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$

$$\boxed{\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)}$$

Definition 1.2 (Product Rule) *The product rule is:*

$$\boxed{\frac{d}{dx}(f(x)g(x)) = f'(x)\cdot g(x) + g'(x)\cdot f(x)}$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

constant mult rule
$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$$
$$\frac{d}{dx}(c) = 0$$
constant

Example 1.3 *Find the derivatives of the following functions:*

(1) $h(x) = e^x\sqrt{5x} = (e^x)\cdot(\sqrt{5x})$

$$h'(x) = \frac{d}{dx}(e^x)\cdot(\sqrt{5}\cdot\sqrt{x}) + e^x\cdot\left(\frac{d}{dx}(\sqrt{5}\cdot x^{1/2})\right)$$
$$= e^x\cdot\sqrt{5}\cdot\sqrt{x} + e^x\cdot\left(\sqrt{5}\cdot\tfrac{1}{2}x^{-1/2}\right)$$
$$= \boxed{e^x\sqrt{5x} + \frac{e^x\sqrt{5}}{2\sqrt{x}}}$$

(2) $f(x) = \frac{\sqrt{2x}}{x^3}$ *Hint: Re-write the fraction as a product*

$\sqrt{2x}\cdot x^{-3}$

1) ~~Quotient Rule~~
2) Product Rule: $\sqrt{2x}\cdot x^{-3}$
3) Power Rule: $\sqrt{2}\, x^{-2.5}$

Power Rule:
$$\boxed{f'(x) = -2.5\sqrt{2}\cdot x^{-3.5}}$$

Product Rule Method:
$$f'(x) = \left(\frac{d}{dx}\sqrt{2x}\right)\cdot x^{-3} + \left(\frac{d}{dx}x^{-3}\right)\cdot\left(\sqrt{2x}\right)$$
$$= \frac{\sqrt{2}}{2\sqrt{x}}\cdot x^{-3} + -3x^{-4}\sqrt{2x}$$
$$= \boxed{\frac{\sqrt{2}}{2x^3\sqrt{x}} + \frac{-3\sqrt{2x}}{x^4}}$$

put into wolfram alpha or desmos to see they're the same

(3) $f(x) = x^2e^x + 3x - 1$

$$\boxed{f'(x) = 2xe^x + x^2e^x + 3}$$

(4) $g(x) = (x^3 - 1)(e^x + \sqrt[4]{x^3})$

this is $x^{15/4}$

If we foil first:
$$g(x) = x^3e^x + x^3x^{3/4} - e^x - x^{3/4}$$
$$g'(x) = 3x^2e^x + x^3e^x + \tfrac{15}{4}x^{11/4} - e^x - \tfrac{3}{4}x^{-1/4}$$
$$\boxed{g'(x) = 3x^2e^x + x^3e^x + \tfrac{15}{4}\sqrt[4]{x^{11}} - e^x - \frac{3}{4\sqrt[4]{x}}}$$

without foiling:
$$g(x) = (x^3-1)(e^x + x^{3/4})$$
$$\boxed{g'(x) = (3x^2)(e^x + x^{3/4}) + (x^3-1)\left(e^x + \tfrac{3}{4}x^{-1/4}\right)}$$

1.2 Quotient Rule

We also need a rule for taking the derivative of functions of the form $\frac{f(x)}{g(x)}$. Here is the rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

(Low)(D high) − (High)(D Low) all over the square of what's below.

Example 1.4 *Find the derivatives of the following functions:*

1. $f(x) = \frac{3x+1}{4x^2+2}$

$$f'(x) = \frac{(4x^2+2)(3) - (3x+1)(8x)}{(4x^2+2)^2}$$

$$f'(x) = \frac{12x^2+6-(24x^2+8x)}{(4x^2+2)^2}$$

$$f'(x) = \frac{-12x^2-8x+6}{(4x^2+2)^2}$$

$$f'(x) = \frac{2(-6x^2-4x+3)}{2^2(2x^2+1)^2}$$

$$\boxed{f'(x) = \frac{-6x^2-4x+3}{2(2x^2+1)^2}}$$

2. $g(t) = \frac{a+b}{c+2e^k}$, *for* a, b, c *real numbers*

1) $g(k) = \frac{a+b}{c+2e^k}$

$$g'(k) = \frac{(c+2e^k)\cdot(0)-(a+b)(2e^k)}{(c+2e^k)^2}$$

$$\boxed{g'(k) = \frac{-(a+b)(2e^k)}{(c+2e^k)^2}} = \frac{-a2e^k-b2e^k}{(c+2e^k)^2}$$

2) $g(t) = \frac{a+b}{c+2e^k}$ a,b,c,k are real #'s

→ just a #, so

$g'(t) = 0$

3. $h(x) = \frac{1-xe^x}{2x+e^x}$ $\frac{f}{g} \to h' = \frac{gf'-fg'}{g^2}$

$f(x) = 1 - xe^x$

$f'(x) = -(e^x + xe^x)$

$g(x) = 2x+e^x$

$g'(x) = 2+e^x$

$$\boxed{h'(x) = \frac{(2x+e^x)(-e^x-xe^x)-(1-xe^x)(2+e^x)}{(2x+e^x)^2}}$$

Example 1.5 *Suppose we have the following information about the differentiable functions f and g:*
$f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, *and* $g'(5) = 7$.
Find the following values:

1. $(fg)'(5)$ product rule; $g \cdot f' + f \cdot g'$ → all at 5:

$$(-3)(6) + (1)(7) = -18 + 7 = -11$$

2. $h'(5)$ *for* $h(x) = 5f(x) - 4g(x)$

$$h'(x) = 5f'(x) - 4g'(x)$$
$$h'(5) = 5f'(5) - 4g'(5)$$
$$= 5 \cdot 6 - 4(7) = 30 - 28 = \boxed{-2}$$

3. $h'(5)$ *for* $h(x) = \frac{g(x)}{1+f(x)}$

$$h'(x) = \frac{(1+f(x)) \cdot g'(x) - g(x)(f'(x))}{(1+f(x))^2}$$
$$h'(5) = \frac{(1+f(5)) \cdot g'(5) - g(5)(f'(5))}{(1+f(5))^2} = \frac{(1+1) \cdot 7 - (-3)(6)}{(1+1)^2} = \frac{14+18}{4} = \frac{32}{4} = \boxed{8}$$

4. $h'(5)$ *for* $h(x) = \frac{g(x)f(x)+1}{g(x)}$

1
6
-3
7

$$h'(x) = \frac{g(x)[g'(x)f(x) + g(x)f'(x)] - [g(x)f(x)+1]g'(x)}{(g(x))^2}$$
$$h'(5) = \frac{g(5)[g'(5)f(5) + g(5)f'(5)] - [g(5)f(5)+1]g'(5)}{(g(5))^2}$$
$$h'(5) = \frac{(-3)[7 \cdot 1 + -3 \cdot 6] - [-3 \cdot 1 + 1] \cdot 7}{(-3)^2}$$
$$= \frac{-3(-11) + 14}{9} = \frac{33+14}{9} = \boxed{\frac{47}{9}}$$