03.02 The Product and Quotient Rules

Sunday, September 20, 2020 4:00 PM

Math 1300: Calculus I **Fall 2020**

Lecture: Section 3.2 The Product and Quotient Rules

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Today's Goal: Deal with derivatives of functions $f(x)g(x)$, $\frac{f(x)}{g(x)}$

Logistics: We will start this on a Wednesday and finish on a Friday. Friday will be a check-in!

Warm-Up 1.1 *True or False:* $\frac{d}{dx}e^x = xe^{x-1}$

variable is in the exponent!

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}b^x = \ln(b)\cdot b^x$$

$$\frac{d}{dx}x^3 = 3x^2$$

variable is base

1.1 Product Rule

We want to be able to take the derivative of products of functions that we can't distribute, for example:

$$h(x) = e^x\sqrt{5x} = e^x \cdot \sqrt{5}\cdot\sqrt{x}$$

don't have a rule yet!

is the product of the two functions:

Our previous rules do not apply! We need a new rule! Let's go back to the definition of derivative and see if we get anywhere:

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

A cool trick that is frequently helpful in math is to add and subtract a term, and see if that helps:

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)]}{h}$$

$$= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Definition 1.2 (Product Rule) *The product rule is:*

$$\frac{d}{dx}(f(x)g(x)) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Example 1.3 *Find the derivatives of the following functions:*

(1) $h(x) = e^x\sqrt{5x} = (e^x)\cdot(\sqrt{5x})$

$h'(x) = \frac{d}{dx}(e^x)\cdot(\sqrt{5}\cdot\sqrt{x}) + e^x\cdot\left(\frac{d}{dx}(\sqrt{5}\cdot x^{1/2})\right)$

$= e^x\cdot\sqrt{5}\cdot\sqrt{x} + e^x\cdot\left(\sqrt{5}\cdot\frac{1}{2}x^{-1/2}\right)$

$= e^x\sqrt{5x} + \frac{e^x\sqrt{5}}{2\sqrt{x}}$

constant mult rule

$\frac{d}{dx}(c\cdot f(x)) = c\cdot\frac{d}{dx}(f(x))$

$\frac{d}{dx}(c) = 0$

constant

(2) $f(x) = \frac{\sqrt{2x}}{x^3}$ *Hint: Re-write the fraction as a product*

$\sqrt{2x}\cdot x^{-3}$

1) Quotient Rule
2) Product Rule: $\sqrt{2x}\cdot x^{-3}$
3) Power Rule: $\sqrt{2}\,x^{-2.5}$

Power Rule:

$f'(x) = -2.5\sqrt{2}\cdot x^{-3.5}$

Product Rule Method:

$f'(x) = \left(\frac{d}{dx}\sqrt{2x}\right)\cdot x^{-3} + \left(\frac{d}{dx}x^{-3}\right)\cdot(\sqrt{2x})$

$= \frac{\sqrt{2}}{2\sqrt{x}}\cdot x^{-3} + -3x^{-4}\sqrt{2x}$

$= \frac{\sqrt{2}}{2x^3\sqrt{x}} + \frac{-3\sqrt{2x}}{x^4}$

put into wolfram alpha or desmos to see they're the same

(3) $f(x) = x^2e^x + 3x - 1$

$f'(x) = 2xe^x + x^2e^x + 3$

(4) $g(x) = (x^3 - 1)(e^x + \sqrt[4]{x^3})$

If we foil first: (this is $x^{15/4}$)

$g(x) = x^3e^x + x^3x^{3/4} - e^x - x^{3/4}$

$g'(x) = 3x^2e^x + x^3e^x + \frac{15}{4}x^{11/4} - e^x - \frac{3}{4}x^{-1/4}$

$g'(x) = 3x^2e^x + x^3e^x + \frac{15}{4}\sqrt[4]{x^{11}} - e^x - \frac{3}{4\sqrt[4]{x}}$

1.2 Quotient Rule

We also need a rule for taking the derivative of functions of the form $\frac{f(x)}{g(x)}$. Here is the rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) = f(x)g'(x)}{[g(x)]^2}$$

Example 1.4 *Find the derivatives of the following functions:*

1. $f(x) = \frac{3x+1}{4x^2+2}$

2. $g(t) = \frac{a+b}{c+2e^k}$, *for* a, b, c *real numbers*

3. $h(x) = \frac{1-xe^x}{2x+e^x}$

Example 1.5 *Suppose we have the following information about the differentiable functions f and g:*
$f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, *and* $g'(5) = 7$.
Find the following values:

1. $(fg)'(5)$

2. $h'(5)$ *for* $h(x) = 5f(x) - 4g(x)$

3. $h'(5)$ *for* $h(x) = \frac{g(x)}{1+f(x)}$

4. $h'(5)$ *for* $h(x) = \frac{g(x)f(x)+1}{g(x)}$