02.08 What does f' say about f?

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Math 1300: Calculus I **Fall 2020**

Lecture 9: Section 2.8: What does f' say about f?

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Today's Goal: What do f', f'' tell us about f?

Logistics: Should be starting this on a Wednesday and finishing it on Friday.
We have a check-in on Friday that will cover 2.6 and 2.7.

Warm-Up 9.1 *True or False: If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.*

False! Corners and cusps can also stop a function from having a derivative i.e., $y = |x|$ at $x = 0$.

9.1 $f'(x)$

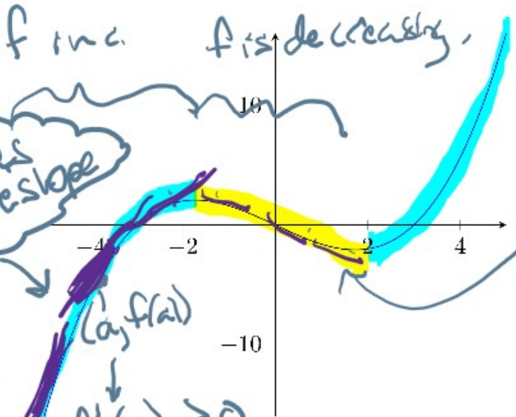
$f'(a)$ specifies the slope of the tangent line to $f(x)$ at $x = a$. If you're trying to "eyeball" the slope of a tangent line, it's hard to tell the difference between a slope of 2 and a slope of 2.5. But it's *not* difficult to tell the difference between a slope of 2 and a slope of -1.

- If $f'(a) > 0$, then $f(x)$ is **increasing** at $x = a$.
- if $f'(x) > 0$ on an interval, then $f(x)$ is **increasing** on that interval.
- If $f'(a) < 0$, then $f(x)$ is **decreasing** at $x = a$.
- if $f'(x) < 0$ on an interval, then $f(x)$ is **decreasing** on that interval.

$\rightsquigarrow f'(-4) > 0$, b/c the graph of f is inc. at $x = -4$.

$f'(x) < 0$ on $(-2, 2)$ b/c f is decreasing on that interval

f inc.   f is decreasing. $y = f(x)$



tangent lines have positive slope

$f'(x) > 0$

$(a, f(a))$

$f'(a) > 0$

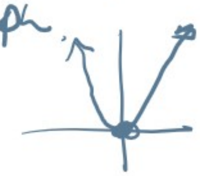
tangent lines have negative slopes

$f'(x) < 0$

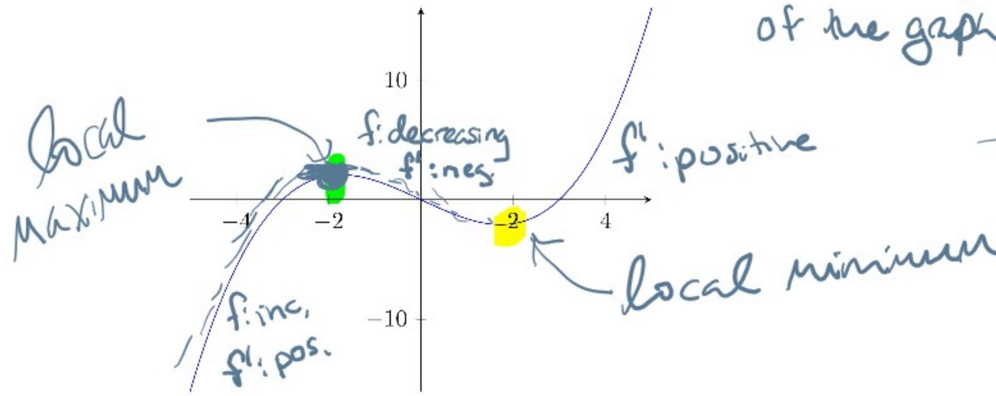
Definition 9.2 *A point where $f'(x) = 0$ or $f'(x)$ DNE is a potential spot where f could be changing from increasing to decreasing or vice versa! An x-value c such that $f'(c) = 0$ or $f'(c)$ DNE is called a* ***critical point****.*

b/c these spots are where f' could change sign!

★ GLOBAL minima & Maxima are the min/max pts of the graph.



9.1.1 Local Minima and Maxima



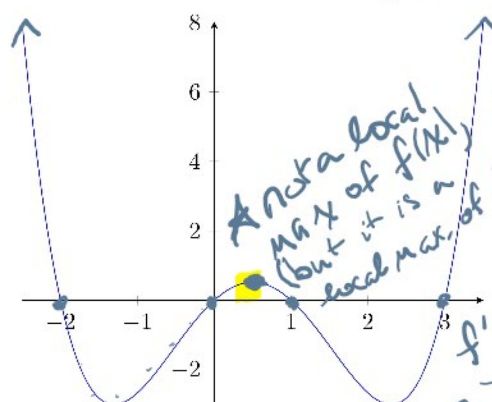
Definition 9.3 *We say that $f(x)$ has a* ***local maximum*** *at $x = c$ if $f(x) \leq f(c)$ for every x in an open interval around $x = c$. This is also referred to as a relative maximum.*

A local maximum can also be characterized by considering $f'(x)$: If $f'(x)$ goes from pos. to neg. at some $x=c$ where $f'(c)=0$, then $(c, f(c))$ is a local maximum.

Definition 9.4 *We say that $f(x)$ has a* ***local minimum*** *at $x = c$ if $f(x) \geq f(c)$ for every x in an open interval around $x = c$. This is also referred to as a relative minimum.*

A local minimum can also be characterized by considering $f'(x)$: If $f'(x)$ goes from neg. to pos. at some $x=c$ w/ $f'(c)=0$, then $(c, f(c))$ is a local min.

Example 9.5 *The graph of $f'(x)$ is shown here.*



★ Put critical values on a # line:

	-2	0	1	3	

f':	+	−	+	−	+
f:	inc.	dec.	inc.	dec.	inc.

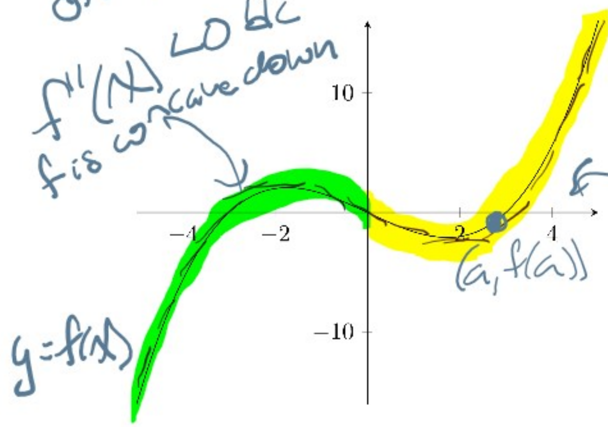
1. *On what interval(s) is $f(x)$ increasing?* $(-\infty,-2)\cup(0,1)\cup(3,+\infty)$
2. *On what interval(s) is $f(x)$ decreasing?* $(-2,0)\cup(1,3)$
3. *At what values of x does $f(x)$ have local maxima and minima?*

$\boxed{x=-2,\ x=1}$ $\boxed{x=0,\ x=3}$

9.2 $f''(x)$ → slopes of tangent lines on $f'(x)$

If the slopes of the tangent lines are increasing (slopes: 1, 1.5, 2, 3, ...) then $f''(x) > 0$.

$f''(x) < 0$ b/c f is concave down



$f''(x) > 0$ b/c f is concave up.

In a previous project, we noticed that $f(x)$ is **concave up** when $f'(x)$ is increasing, and $f(x)$ is **concave down** when $f'(x)$ is decreasing. We can also characterize this using $f''(x)$:

- If $f''(a) > 0$, then $f(x)$ is **concave up** at $x = a$.
- If $f''(x) > 0$ on an interval, then $f(x)$ is **concave up** on that interval. f is concave up on $(0, \infty)$
- If $f''(a) < 0$, then $f(x)$ is **concave down** at $x = a$.
- If $f''(x) < 0$ on an interval, then $f(x)$ is **concave down** on that interval.

Remark 9.6 *$f''(x) > 0$ is saying the same thing as '$f'(x)$ is increasing'!* ⟷ f is concave up
Likewise, $f''(x) < 0$ is the same thing as saying '$f'(x)$ is decreasing'! ⟷ f is concave down.

Definition 9.7 *An **inflection point** is a point (x, y) where the function changes concavity.* need to change concavity

Inflection points can only occur at x-values where $f''(x) = 0$ or DNE, but we don't use the word inflection point unless there is a *change* in concavity. For example, if $f''(x)$ goes from positive, to zero, to positive – there is no change in concavity, so this is not an inflection point. However, if $f''(x)$ goes from positive to zero to negative, then we have passed through an inflection point. In addition, it needs to be a *point* on the graph of $f(x)$, so the x-value of the location of change in concavity must be in the domain of $f(x)$.

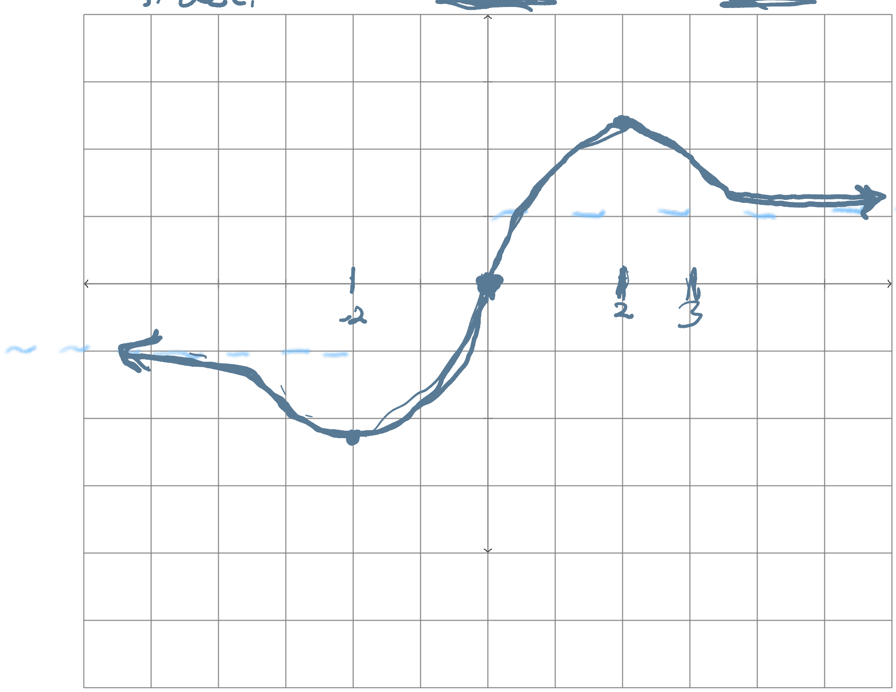
i.e, f'' must change sign!

Example 9.8 (Example due to Dr. Patrick Newberry) *Sketch a graph with the given properties:*

- $f'(x) > 0$ *for* $|x| < 2$ ⇝ f is inc. on $(-2, 2)$
- $f'(x) < 0$ *for* $|x| > 2$ ⇝ f is dec. on $(-\infty, -2) \cup (2, \infty)$
- $f'(2) = 0$ ⇝ f has a horiz. tangent line; f has local max at $x = 2$
- $f''(x) < 0$ *for* $0 < x < 3$ ⇝ f is concave down on $(0, 3)$
- $\lim_{x\to\infty} f(x) = 1$ HA on right @ $y = 1$
- $f(-x) = -f(x)$ *for all* x odd sym.: graph looks same if you rotate 180° * $f(0) = 0$

Hint: Translate each of these bullet points to a statement about f. Use sign lines for $f'(x), f''(x)$ to keep track of the information.

f: dec. f: inc. f: concave down f: dec.



~~$f(0) = 1$;~~
~~$f(-0) = -f(0)$~~
~~$1 = -1$~~
not possible!

9.3 Antiderivatives

$$\begin{tikzcd} f(x) \arrow[r, bend left, "\text{take the deriv.}"] & f'(x) \arrow[l, bend left, "?"] \end{tikzcd}$$

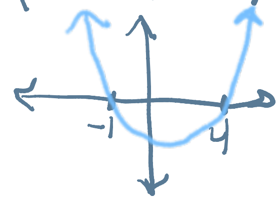
Definition 9.9 *A function $F(x)$ is an* ***antiderivative*** *of a function $f(x)$ if $F'(x) = f(x)$.*

Big idea: If we are given some function which we *know* is a derivative, can we recover the original function? How close can we get?

Example 9.10 *Suppose $f(x) = x^2 - 3x - 4$, and $F(x)$ is an antiderivative of $f(x)$.* : $F'(x) = f(x)$

$$\boxed{F'(x) = x^2 - 3x - 4}$$

$$(x-4)(x+1)$$



1. *On what interval(s) is $F(x)$ increasing?*

   $F'(x) > 0$

   same

   $f(x) > 0$

   $$\boxed{(-\infty, -1) \cup (4, \infty)}$$

2. *On what interval(s) is $F(x)$ decreasing?*

   $$\boxed{(-1, 4)}$$

3. *At what x-values does $F(x)$ have any inflection points, if any?*

   $F''(x)$ changes sign

   take derivative of $f(x) = F'(x)$

   $$F''(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
   $$= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - 4 - (x^2 - 3x - 4)}{h}$$
   $$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - 4 - x^2 + 3x + 4}{h}$$
   $$= \lim_{h \to 0} \frac{h^2 + 2xh - 3h}{h}$$
   $$= \lim_{h \to 0} (h + 2x - 3) = 2x - 3$$

   $2x - 3 > 0$ for $x > \frac{3}{2}$

   and $2x - 3 < 0$ for $x < \frac{3}{2}$

   so F has an inflection point @ x

4. *What is $F(0)$?*

   F' is not enough to give us info on the values of F