02.04 Continuity

Tuesday, September 1, 2020 12:29 PM

Math 1300: Calculus I **Fall 2020**

Lecture 5: Section 2.4: Continuity

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Today's Goal: Learn the definition of a continuous function, and some important theorems of continuo

Logistics:

Check-in Friday, Written Homework due Thursday by 6pm

We are starting this section Wednesday (possibly Tuesday?), but we will not finish it until Friday.

Warm-Up 5.1 *Evaluate* $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}\cdot\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-\sqrt{x}+\sqrt{x}-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1}(\sqrt{x}+1) = \boxed{2}$$

5.1 Definition

Definition 5.2 *A function $f(x)$ is continuous at a value $x = a$ if*

$$\boxed{\lim_{x\to a} f(x) = f(a).}$$

$f(x)=\sqrt{x}$ is continuous at $x=1$

There is a lot hidden in this definition!

- $\boxed{\lim_{x\to a} f(x) \text{ exists!}}$ We need the left and right limits to agree for this to be true.
- $f(a)$ needs to exist! a needs to be in the domain of the function for this to happen.
- The above two quantities need to be equal.

If $f(x)$ does not satisfy this definition at the value a, we say $f(x)$ **is discontinuous at** a, or $f(x)$ **has a discontinuity at** a.

$f(x) = [[x]]$ ~ "rounding down function"

has a discontinuity at any integer

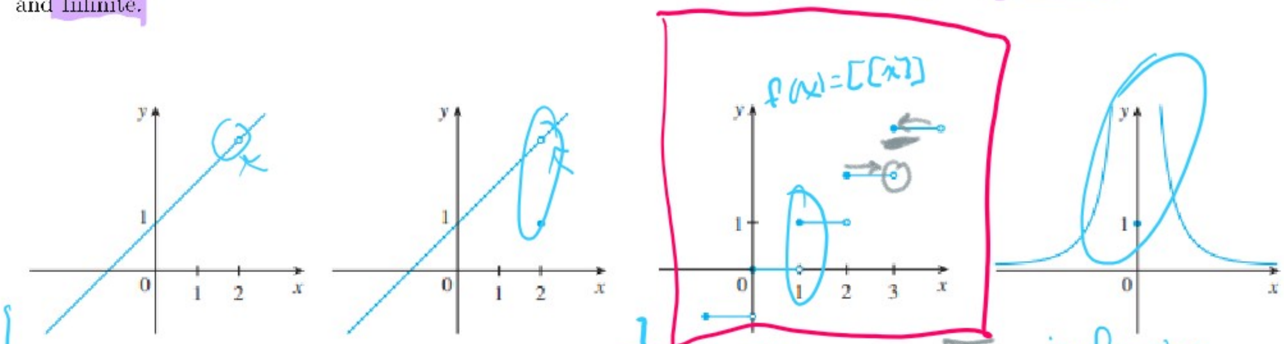
$[[2.1]] = 2$

$[[2]] = 2$

$[[1.9]] = 1$

5.2 Discontinuities: From a Graph

Where are the following functions continuous? Where are they discontinuous? Can we classify the types of discontinuities these different graphs display? There are three kinds of discontinuity: Jump, Removeable, and Infinite.



Removeable, hole

Jump

infinite

Continuous functions have graphs you can draw without picking up your pencil

$[[x]]$ is not continuous at $x = 3$

$[[x]]$ is "continuous on the right" at $x = 3$

$\lim_{x \to 3^-} [[x]] = 2$

$\lim_{x \to 3^+} [[x]] = 3$

$[[3]] = 3$

$[[x]]$ is discontinuous at every integer

$[[x]]$ is cont. on $\mathbb{R} \setminus \mathbb{Z}$

real #'s — "without" — integers

5.3 Discontinuities: From an Equation

5.3.1 Domain

This will have a similar feel to finding the domain of a function given by an equation. Recall, you need to check things like **denominators** and **even radicals** (square roots, 4th roots, etc.).

Example 5.3 *Find the domain of the function:*

use interval notation

$$r(t) = \frac{(\sqrt{t+1})(\sqrt[3]{t-6})}{t^2+3t-4}$$

Denom: can't be 0.

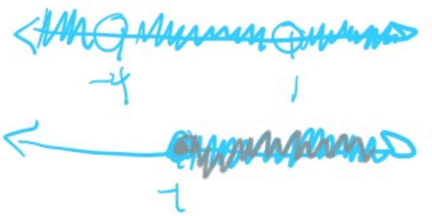
$(t+4)(t-1)$

$t \neq -4, 1$

Can't sq. root negative #'s:

$t+1 \geq 0$

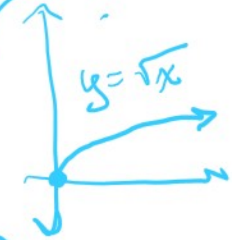
$t \geq -1$



both

include exclude

$[-1, 1) \cup (1, \infty)$



5.3.2 Discontinuities

Example 5.4 *Consider the function:*

$$\frac{(x+1)(x-1)}{(x+2)(x^2-2x+4)}$$

has a vertical Asym. @ $x=-2$

$$h(x) = \begin{cases} \frac{x^2-1}{x^3+8} & , \textit{ if } x \neq -2 \\ 4 & , \textit{ if } x = -2 \end{cases}$$

Where is $h(x)$ discontinuous? Where is $h(x)$ continuous? Use interval notation to answer.

at $x = -2$

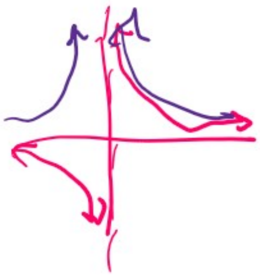
$(-\infty, -2) \cup (-2, \infty)$

Example 5.5 *Where is the following function continuous?*

$$h(x) = \frac{1}{x} = 1 \div x \text{ can't divide by } 0$$

discontinuous at $x=0$: $\lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty$ (think: plug in $x=-.001, ..$)

$\lim_{x\to 0^+}\left(\frac{1}{x}\right) = +\infty$ $\boxed{\lim_{x\to 0}\left(\frac{1}{x}\right) \text{ DNE}}$



cont: $\boxed{(-\infty,0)\cup(0,\infty)}$ ☆

Example 5.6 *Where is the following function continuous?*

$$f(x) = \begin{cases} \frac{x^2-x-6}{x+2}, & \text{if } x \neq -2 \\ -5, & \text{if } x = -2 \end{cases}$$

$$\frac{x^2-x-6}{x+2} = \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}}$$

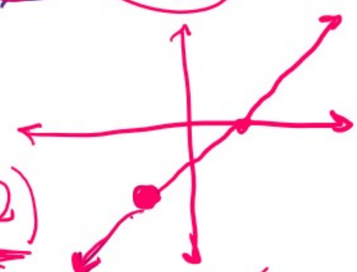
cancelling means this rational function has a hole, not a VA at $x=-2$

$x-3$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2}\frac{x^2-x-6}{x+2}$$

$$= \lim_{x\to -2}(x-3) \text{ ☆}$$

$$y = \frac{x^2-x-6}{x+2} = -5 = f(-2)$$



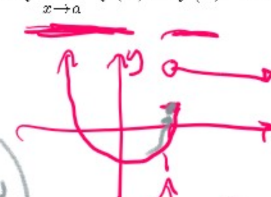
This function is cont. $(-\infty,\infty)$

Definition 5.7 *We say that a function is* **continuous on the left at a** *if* $\lim_{x\to a^-} f(x) = f(a)$. *We likewise define being continuous on the right.*

Example 5.8 *Is* $[[x]]$ *continuous on the left and/or right at* $a = 5$?

"round down to nearest integer"

(did for $a=3$ 2 pages above)



this function is not cont. at 1, but it is cont. on the left at 1.

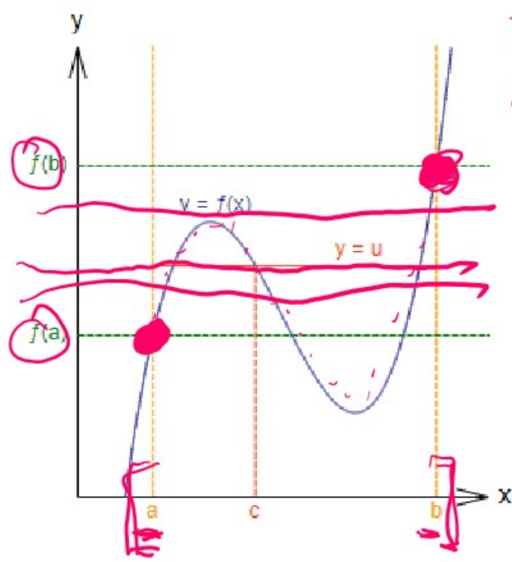
Ex: Is x^3-3x+2^x continuous?
∴ yes, it's a polynomial w/ an exponential function and those are both cont.

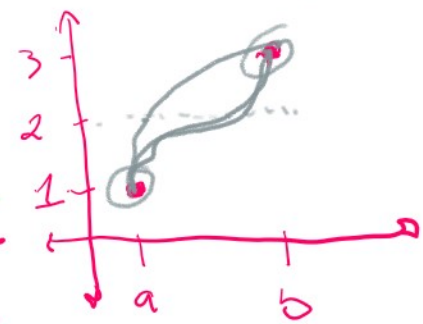
Theorem 5.9 *The following types of functions are continuous at every value in their domain:*

↑ don't pick up pencil to draw

- *polynomials* ✓
- *rational functions* : only in domain. careful of holes and VA's.
- *root functions* — $\sqrt{x}$:
- *trig functions* : some have VA's
- *exponential functions* — 2^x:
- *log functions* $\ln(x)$: has a VA @ $x=0$, but it is cont. in its domain ✓

Theorem 5.10 (The Intermediate Value Theorem) *Suppose f is continuous on the closed interval $[a,b]$ (of x-values). If u is any number between $f(a)$ and $f(b)$, then there exists an x-value c in (a,b) such that $f(c)=u$.*





(Image from Kpengboy / Public domain on the Wikipedia article for Intermediate Value Theorem)

show there exists a "c" btwn "a=0" and "b=1" such that $f(c)=0$.

$[a,b]$

Example 5.11 *Use the intermediate value theorem to show that the function $f(x)$ as below has a root between $x = 0$ and $x = 1$:*

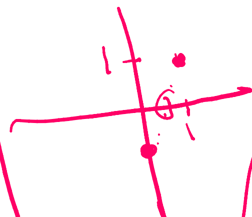
$$f(x) = \sqrt[3]{x} + x - 1$$

f is continuous on $[0,1]$ b/c it's a radical function added w/ a polynomial function - it's cont. on its domain $(-\infty,\infty)$

$f(0) = -1$

$f(1) = 1$

values at endpoints



By IVT since 0 is between $f(0)=-1$ and $f(1)=1$, the function has a value c in $(0,1)$ such that $f(c)=0$.

steps

1. show f is cont. on $[a,b]$ (so IVT applies)
2. Look at values @ endpoints $f(a)$, $f(b)$
3. Make conclusion sentence.

Example 5.12 *Use the intermediate value theorem to show that the following equation has a solution in the interval* $(1, 2)$.

$$\sin(x) = x^2 - x$$

This is equivalent to saying "use the IVT to show that the function $f(x) = \sin(x) - (x^2 - x)$ has a zero between $x=1$ and $x=2$".
This is an easier form to answer!

① $f(x) = \sin(x) - (x^2 - x)$ is cont. on $(-\infty,\infty)$, so it is certainly continuous on the closed interval $[1,2]$

② $f(1) = \sin(1) - (1^2 - 1) = \sin(1) \approx .84\ldots > 0$
$f(2) = \sin(2) - (2^2 - 2) = \sin(2) - 2 \approx -1.09 < 0$
*0 is inbetween $f(1)$ and $f(2)$

With ① and ②, we can use the IVT to conclude that there is some $c \in (1,2)$ such that $f(c) = 0$ ✓