02.03 Limit Laws

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Math 1300: Calculus I **Fall 2020**

Lecture 4: Section 2.3: Calculating Limits Using the Limit Laws

Lecturer: Sarah Arpin

Today's Goal: Learn more techniques for evaluating limits.

Logistics: First evening quiz is Tuesday night. Make sure you have the proctorio extension in your chrome browser, and that you have a webcam. If either of these are issues, please contact me.

Warm-Up 4.1 *True or False: If both the left and right-hand limits of $f(x)$ as x approaches a exist, then $\lim_{x\to a} f(x)$ always exists.* false! Because the left and right limits could both exist but equal different #'s

4.1 Limit Laws

putting these all together: If f(x) is a polynomial or a rational function w/ "a" in the domain, then: $\lim_{x\to a}(f(x)) = f(a)$

☆ you can just plug in!

- $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ Sum
- $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ Difference
- $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$, for any constant c (think: 2, pi, -233, etc.) constant multiple
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$ product
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$. quotient
- $\lim_{x\to a} f(x)^n = (\lim_{x\to a} f(x))^n$ power
- $\lim_{x\to a} c = c$, for any constant c (again, this: $2, \pi, -233$, etc.) Constant
- $\lim_{x\to a} x = a$. identity
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ radical

$\lim_{x\to 2}(x) = 2.$

So: $\lim_{x\to 2}(\pi x) = \pi \lim_{x\to 2}(x) = \boxed{\pi\cdot 2}$ ✓

Example 4.2 *Evaluate the following limits:*

(a) $\lim_{x\to -1} \frac{2x^2-3x}{x^2+1}$ Check that $\lim_{x\to -1}(x^2+1) \neq 0$. Then we plug in!

(b) $\lim_{x\to 2}(3x^3-4x^2+4)$ A polynomial! We can plug in

(c) $\lim_{x\to 0} g(x)$ *where* $g(x) = \begin{cases} \sin(x) & \text{, if } x\neq 0 \\ 2 & \text{, if } x = 0 \end{cases}$

(d) $\lim_{x\to 0} |x|$

Answers:
(a) 5/2
(b) 12
(c) 0
(d) 0

4.2 Algebraic Techniques

4.2.1 Factoring Rational Functions

We have seen a lot of polynomial and rational function limits that basically just amount to plugging in the value, after we use limit laws:

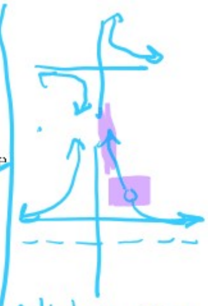
Check: x^2+2x @ $x=1$: $1+2=3\neq 0$ ✓

$\lim_{x\to 1} \frac{x^2-4}{x^2+2x} =$ $\frac{1-4}{3} = \frac{-3}{3} = \boxed{-1}$

But it does not always work:

check: x^2+2x @ $x=-2$: $4-4=0$:(

$\lim_{x\to -2} \frac{x^2-4}{x^2+2x} =$



★Holes vs. asymptotes!

We would like to find a way for this to work, and factoring can sometimes help.

In pre-calculus, you probably learned about simplifying rational expressions such as:

$$\frac{x^2-1}{x^2+2x-3} \to \boxed{\frac{(x+1)(x-1)}{(x-1)(x+3)}} \to \frac{x+1}{x+3}$$

★away from $x\neq 1$ ★

$\frac{0}{5}=0$

$\frac{0}{0}$:(

The last step is only true for values of x that don't make the cancelled factor 0! Basically, you are only allowed to cancel expressions like $\frac{2}{2}=\frac{1}{1}$, but the expression $\frac{0}{0}$ is undefined and cannot be cancelled.

We can get around this issue using limits! Remember that $\lim_{x\to a} f(x)$ never actually uses the value $f(a)$, it's just telling us what's happening as x approaches a. **This means that cancelling a factor of $x-a$ from numerator and denominator would be allowed!** Let's see an example:

Example 4.3 *Evaluate* $\lim_{x\to -2} \frac{x^2-4}{x^2+2x}$, *writing a small justification for each step you take.* ★Rational Functions

$$\lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{x\cancel{(x+2)}} = \lim_{x\to -2} \frac{x-2}{x}$$

Check: $x\neq 0$ @ -2 ✓ $-2\neq 0$

$$= \frac{-2-2}{-2} = \frac{-4}{-2} = \boxed{2}$$

There are other algebraic techniques, such as combining rational expressions with common denominators and simplifying complex fractions. To summarize, it is best to **simplify an expression** and then evaluate the limit using limit laws.

Example 4.4 *Evaluate the following limits:*

1. $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)} = \lim_{x\to 4} \frac{x}{x+1} = \boxed{\frac{4}{5}}$

2. $\lim_{y\to 0} \frac{(2+y)^3-8}{y}$

$(2+y)(2+y)(2+y)$

$= \lim_{y\to 0} \frac{(\cancel{8}+12y+6y^2+y^3)\cancel{-8}}{y} = \lim_{y\to 0} \frac{\cancel{y}(12+6y+y^2)}{\cancel{y}}$

$= \lim_{y\to 0} (12+6y+y^2) = \boxed{12}$

3. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+1}\right) = \lim_{x\to 0} \left(\frac{x^2+1}{x(x^2+1)} - \frac{x}{x(x^2+1)}\right) = \lim_{x\to 0} \frac{x^2-x+1}{x(x^2+1)}$

* zeros in denom that don't cancel give us vertical asymptotes. use numerical values to see what kind of VA.

$\boxed{\text{Does not exist}}$ $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$

4. $\lim_{t\to -4} \frac{\sqrt{t^2-9}-5}{t+4}$

$\cancel{\sqrt{t^2+9} = \sqrt{t^2} + \sqrt{9}}$

don't distribute radicals!!!

$2(t^2+9) = 2t^2+18$

Rationalize the numerator: $\frac{(\sqrt{t^2+9}-5)}{(t+4)} \cdot \frac{(\sqrt{t^2+9}+5)}{(\sqrt{t^2+9}+5)} = \frac{t^2+9\cancel{+5\sqrt{t^2+9}}\cancel{-5\sqrt{t^2+9}}-25}{(t+4)(\sqrt{t^2+9}+5)}$

$= \frac{t^2-16}{\text{denom}} = \frac{\cancel{(t+4)}(t-4)}{\cancel{(t+4)}(\sqrt{t^2+9}+5)}$

$= \lim_{t\to -4} \frac{t-4}{\sqrt{t^2+9}+5} = \frac{-4-4}{\sqrt{16+9}+5} = \frac{-8}{10} = \boxed{\frac{-4}{5}}$

b/c $|x-1| = x-1$ for $x \geq 1$

Example 4.5 *Evaluate the following limits:*

(A) $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|}$ $= \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1^+} x+1 = \boxed{2}$

(B) $\lim_{x\to 1^-} \frac{x^2-1}{|x-1|}$ $= \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} = \lim_{x\to 1^-} \frac{(x+1)\cancel{(x-1)}}{-\cancel{(x-1)}} = \lim_{x\to 1^-} \frac{x+1}{-1} = -2$

because $|x-1| = -(x-1)$ for $x \leq 1$!!!

(C) $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ DNE! left and right limits disagree!

(D) *Double brackets* $[[x]]$ *denotes the greatest integer function - it will always round down to the nearest integer. With that in mind, evaluate:*

$\lim_{x\to 2^-} [[x]]$

$\lim_{x\to 2^+} [[x]]$

$\lim_{x\to 0^+} [[x-1]]$

$\lim_{x\to 2^-} [[x]] = 1$ (b/c 1.9, 1.999, ... will always round down)

$\lim_{x\to 2^+} [[x]] = 2$ (plug in 2.1, 2.0001, etc.)

$\lim_{x\to 0^+} [[x-1]] = -1$ (plug in 0.1, 0.01, etc)