Math 1300: Calculus I **Fall 2020**

Lecture 4: Section 2.3: Calculating Limits Using the Limit Laws

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Today's Goal: Learn more techniques for evaluating limits.

Logistics: First evening quiz is Tuesday night. Make sure you have the proctorio extension in your chrome browser, and that you have a webcam. If either of these are issues, please contact me.

Warm-Up 4.1 *True or False: If both the left and right-hand limits of $f(x)$ as x approaches a exist, then $\lim_{x\to a} f(x)$ always exists.*

4.1 Limit Laws

- $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$, for any constant c (think: 2, pi, -233, etc.)
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$.
- $\lim_{x\to a} f(x)^n = (\lim_{x\to a} f(x))^n$
- $\lim_{x\to a} c = c$, for any constant c (again, this: $2, \pi, -233$, etc.)
- $\lim_{x\to a} x = a$.
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Example 4.2 *Evaluate the following limits:*

(a) $\lim_{x\to -1} \frac{2x^2-3x}{x^2+1}$

(b) $\lim_{x\to 2}(3x^3-4x^2+4)$

(c) $\lim_{x\to 0} g(x)$ *where* $g(x) = \begin{cases} \sin(x) & , \text{ if } x\neq 0 \\ 2 & , \text{ if } x = 0 \end{cases}$

(d) $\lim_{x\to 0}|x|$

4.2 Algebraic Techniques

4.2.1 Factoring Rational Functions

We have seen a lot of polynomial and rational function limits that basically just amount to plugging in the value, after we use limit laws:
$\lim\limits_{x\to 1} \frac{x^2-4}{x^2+2x} =$

But it does not always work:
$\lim\limits_{x\to -2} \frac{x^2-4}{x^2+2x} =$

We would like to find a way for this to work, and factoring can sometimes help.

In pre-calculus, you probably learned about simplifying rational expressions such as:

$$\frac{x^2-1}{x^2+2x-3} \to \frac{(x+1)(x-1)}{(x-1)(x+3)} \to \frac{x+1}{x+3}$$

The last step is only true for values of x that don't make the cancelled factor 0! Basically, you are only allowed to cancel expressions like $\frac{2}{2} = \frac{1}{1}$, but the expression $\frac{0}{0}$ is undefined and cannot be cancelled.

We can get around this issue using limits! Remember that $\lim\limits_{x\to a} f(x)$ never actually uses the value $f(a)$, it's just telling us what's happening as x approaches a. **This means that cancelling a factor of $x-a$ from numerator and denominator would be allowed!** Let's see an example:

Example 4.3 *Evaluate $\lim\limits_{x\to -2} \frac{x^2-4}{x^2+2x}$, writing a small justification for each step you take.*

There are other algebraic techniques, such as combining rational expressions with common denominators and simplifying complex fractions. To summarize, it is best to **simplify an expression** and then evaluate the limit using limit laws.

Example 4.4 *Evaluate the following limits:*

1. $\lim\limits_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$

2. $\lim\limits_{y \to 0} \frac{(2+y)^3-8}{y}$

3. $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2+1} \right)$

4. $\lim\limits_{t \to -4} \frac{\sqrt{t^2+9}-5}{t+4}$

Example 4.5 *Evaluate the following limits:*

(A) $\lim_{x \to 1^+} \frac{x^2-1}{|x-1|}$

(B) $\lim_{x \to 1^-} \frac{x^2-1}{|x-1|}$

(C) $\lim_{x \to 1} \frac{x^2-1}{|x-1|}$

(D) Double brackets $[[x]]$ *denotes the greatest integer function - it will always round down to the nearest integer. With that in mind, evaluate:*

$\lim_{x \to 2^-} [[x]]$

$\lim_{x \to 2^+} [[x]]$

$\lim_{x \to 0^+} [[x-1]]$