Super-Apollonian Continued Fractions

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Continued Fractions on the line

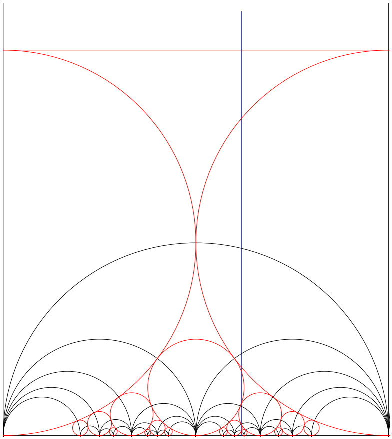


Figure: Approximating $\frac{\sqrt{5}-1}{2} = [0; \bar{1}]$

A simple continued fraction $x = [a_0; a_1, a_2, \dots]$ records the sequence of triangles through which the geodesic $\overrightarrow{\infty x}$ passes, the digit a_i recording the number of triangles through which $\overrightarrow{\infty x}$ passes before switching "sides" of the triangle, and the convergents $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ recording the sequence of horoball neighborhoods (red) through which $\overrightarrow{\infty x}$ passes.

Reflective Continued Fractions

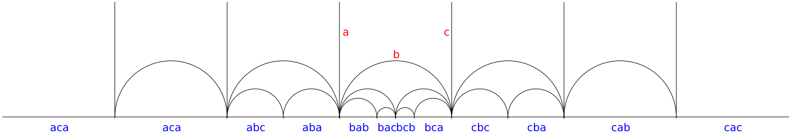
Consider the group $\Gamma = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \rangle \subseteq PGL_2(\mathbb{R}) \cong \mathrm{Isom}(H^2)$, generated by reflections in the walls of the ideal hyperbolic triangle with vertices $\{0, 1, \infty\}$:

$$\mathfrak{a}(x) = -x, \ \mathfrak{b}(x) = \frac{x}{2x-1}, \ \mathfrak{c}(x) = 2 - x.$$

We have

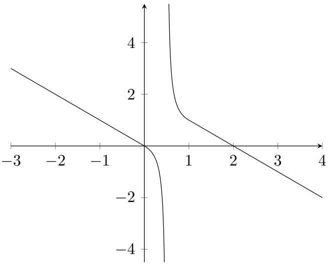
$$1 \to \Gamma \to PGL_2(\mathbb{Z}) \to PGL_2(\mathbb{Z}/(2)) \to 1,$$
$$PGL_2(\mathbb{Z}) = \Gamma \rtimes S_3, \ \Gamma \cong \mathbb{Z}/(2) * \mathbb{Z}/(2) * \mathbb{Z}/(2).$$

Words in these generators index the triangles in the tessellation, and the words of length n partition the line into $3 \cdot 2^{n-1}$ subintervals. The partition by words of length $m > n$ refines the partition by words of length n. Irrational x are then uniquely coordinatized by infinite words in the alphabet $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$.



Dynamical System on the Line

$$T(x) = \begin{cases} \mathfrak{a}(x) = -x & x \in [-\infty, 0] \\ \mathfrak{b}(x) = \frac{x}{2x-1} & x \in [0,1] \\ \mathfrak{c}(x) = 2 - x & x \in [1, \infty] \end{cases}$$



The expansion of x as an infinite word in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is produced by a dynamical system $T : P^1(\mathbb{R}) \to P^1(\mathbb{R})$.

There are three neutral fixed points, $\frac{0}{1}$, $\frac{1}{1}$, and $\frac{1}{0}$, to which rational points descend in finitely many steps depending on the parity of the numerator and denominator (even/odd, odd/odd, odd/even).

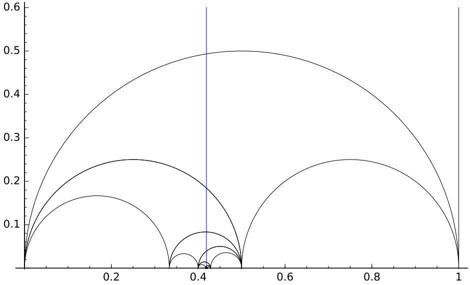
If x is irrational and $\mathfrak{m} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is defined by $T^n x = \mathfrak{m}(T^{n-1}x)$, then we have three sequences of rational convergents

$$\lim_{n\to\infty} \mathfrak{m}_1 \mathfrak{m}_2 \dots \mathfrak{m}_n x_0 = x, \;\; x_0 = 0, 1, \infty,$$

the vertices of the triangles through which the geodesic $\overrightarrow{\infty x}$ passes.

These approximations can also be constructed as a sequence of mediants starting with $(\frac{1}{1}, \frac{1}{0}, \frac{0}{1})$ or $(\frac{1}{1}, \frac{-1}{0}, \frac{0}{1})$.

Example



Here is a random number

$x = 0.4189513796210592\ldots$

with expansion

$\mathfrak{x} = \mathfrak{bacabcacbcacacababac}\ldots$

The first 20 convergents are ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ updating positions 1, 2, 3, mediant in red)

$(1/1, 1/0, 0/1):$

$(1/1, 1/2, 0/1),\ (1/3, 1/2, 0/1),\ (1/3, 1/2, 2/5),$

$(3/7, 1/2, 2/5),\ (3/7, 5/12, 2/5),\ (3/7, 5/12, 8/19),$

$(13/31, 5/12, 8/19),\ (13/31, 5/12, 18/43),\ (13/31, 31/74, 18/43),$

$(13/31, 31/74, 44/105),\ (75/179, 31/74, 44/105),\ (75/179, 31/74, 106/253),$

$(137/327, 31/74, 106/253),\ (137/327, 31/74, 168/401),\ (199/475, 31/74, 168/401),$

$(199/475, 367/876, 168/401),\ (535/1277, 367/876, 168/401),\ (535/1277, 703/1678, 168/401),$

$(871/2079, 703/1678, 168/401),\ (871/2079, 703/1678, 1574/3757)$

$= (0.4189514\ldots, 0.4189511\ldots, 0.4189512\ldots).$

Invertible Extension

The invertible extension $\widetilde{T}$ of T is defined on the space of geodesics $\mathcal{G}$ that intersect the fundamental triangle, acting on $\overrightarrow{yx}$ depending on x:

$$\widetilde{T}(y,x) = \begin{cases} (\mathfrak{a}(y), \mathfrak{a}(x)) & x \in [-\infty, 0] \\ (\mathfrak{b}(y), \mathfrak{b}(x)) & x \in [0, 1] \\ (\mathfrak{c}(y), \mathfrak{c}(x)) & x \in [1, \infty] \end{cases}$$

$\widetilde{T}$ associates to the geodesic $\overrightarrow{yx}$ a bi-infinite word $\mathfrak{y}^{-1}\mathfrak{x}$ in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, which we will relate to the geodesic flow in H^2/Γ.

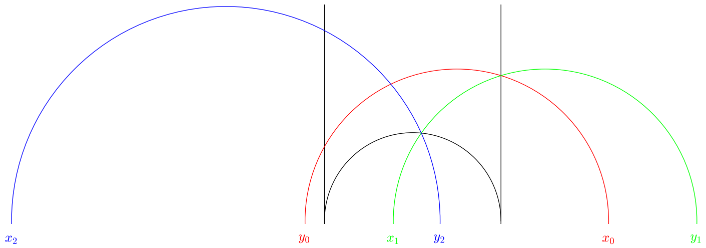
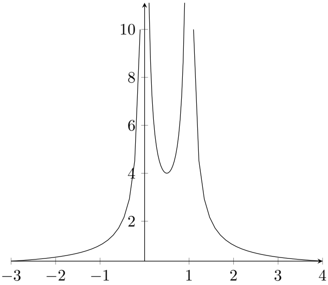


Figure: Two iterations of $\widetilde{T}$, red to green to blue

Invariant Measure

$$d\mu(x) = \begin{cases} \frac{dx}{-x} & x < 0, \\ \frac{dx}{x(1-x)} & 0 < x < 1, \\ \frac{dx}{x-1} & x > 1, \end{cases}$$

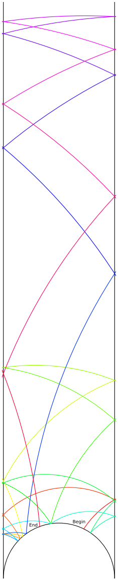


The measure $d\eta(y,x) = (x-y)^{-2}dxdy$ is isometry-invariant on the space of geodesics in the hyperbolic plane.

Since $\widetilde{T}$ is a bijection defined piecewise by isometries, $\eta|_{\mathcal{G}}$ is $\widetilde{T}$-invariant.

Pushing forward to the second coordinate gives an **infinite** T-invariant measure μ. We will see that $(\mathcal{G}, \widetilde{T}, \eta|_{\mathcal{G}})$ is ergodic (hence also the ergodicity of $(P^1(\mathbb{R}), T, \mu)$).

Cross-section of the Geodesic Flow



The word $\mathfrak{y}^{-1}\mathfrak{x}$ associated to the geodesic $\overrightarrow{yx}$ records the sequence of collisions with the walls of the triangle in H^2/Γ, and $\widetilde{T}$ is the first-return of the geodesic flow (billiards in the triangle) to the cross-section defined by points/directions on the boundary. The return time is integrable with respect to $d\eta(y,x)$.

[For instance, a geodesic $(y,x) \in [-\infty, 0] \times [1, \infty]$, has retrun time

$$r(y,x) = \frac{1}{2} \log \left(\frac{x(1-y)}{y(1-x)} \right).$$

and the integral is

$$\frac{1}{2} \int_{-\infty}^{0} \int_{1}^{\infty} \log \left(\frac{x(1-y)}{y(1-x)} \right) \frac{dx dy}{(y-x)^2} = \frac{\pi^2}{6}.]$$

Moving up a dimension...

The system just discussed is the restriction of a pair of dynamical systems on the plane to the real line, which we now describe.

Descartes Quadruples

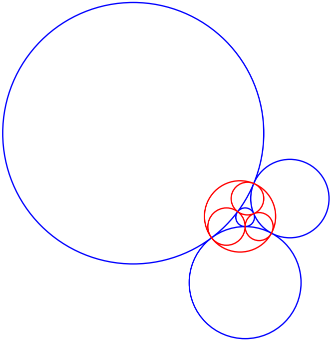


Figure: A Descartes quadruple and its dual

A *Descartes quadruple* is a collection of four mutually tangent circles in the plane (ordered and oriented so that the interiors do not overlap), and its dual quadruple consists of the four mutually tangent circles passing orthogonally through its points of tangency. The curvatures (oriented inverse radii) of the circles satisfy

$$2(c_1^2+c_2^2+c_3^2+c_4^2) = (c_1+c_2+c_3+c_4)^2,$$

i.e. they are zeros of the quadratic form

$$D(x) = x \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} x^t$$

Descartes Quadruples (cont.)

We can move between "adjacent" quadruples with four *swaps* (fix three circles and replace the fourth by its inversion in the disjoint dual circle) and four *inversions* (fix one circle and replace the other three with their inversions in the fixed circle).

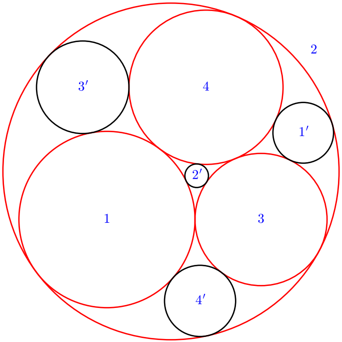


Figure: The four "swaps" S_i

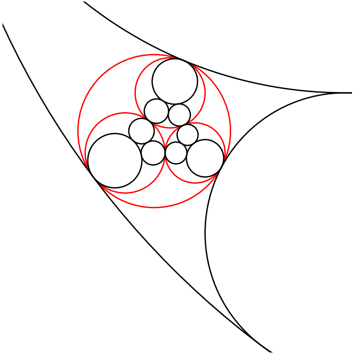


Figure: The four "inversions" $S_i^{\perp}$

The Super-Apollonian Group

The *super-Apollonian group* $\mathcal{A}^S \subseteq O_D(\mathbb{Z}) \subseteq GL_4(\mathbb{Z})$ is the group generated by the swaps and inversions (acting on Descartes quadruples, circles coordinatized by indefinite binary hermitian forms)

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix},$$

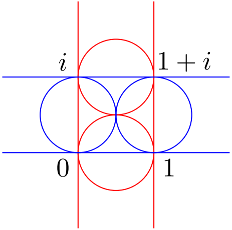
$$S_i^{\perp} := S_i^t.$$

$\mathcal{A}^S$ is a **right-angled** hyperbolic Coxeter group with presentation

$$\langle S_i, S_i^{\perp},\ 1 \le i \le 4 : S_i^2 = (S_i^{\perp})^2 = [S_i, S_j^{\perp}] = 1,\ i \neq j \rangle.$$

Octahedral Reflection Group

We will work with the following group of Möbius transformations isomorphic to $\mathcal{A}^S$, **reflections in the sides of the (finite volume) ideal right-angled octahedron** with vertices $\{0, 1, \infty, i, 1+i, 1/(1-i)\}$ (the intersection of two infinite volume hyper-ideal tetrahedra, boundary circles in red/blue)



$$\Gamma = \langle \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_1^\perp, \mathfrak{s}_2^\perp, \mathfrak{s}_3^\perp, \mathfrak{s}_4^\perp \rangle \subseteq PSL_2(\mathbb{C}) \rtimes \langle \bar{z} \rangle \cong \mathrm{Isom}(H^3),$$

$$\mathfrak{s}_1 = \frac{(1+2i)\bar{z} - 2}{2\bar{z} - 1 + 2i}, \ \mathfrak{s}_2 = \frac{\bar{z}}{2\bar{z} - 1}, \ \mathfrak{s}_3 = -\bar{z} + 2, \ \mathfrak{s}_4 = -\bar{z},$$

$$\mathfrak{s}_1^\perp = \bar{z}, \ \mathfrak{s}_2^\perp = \bar{z} + 2i, \ \mathfrak{s}_3^\perp = \frac{\bar{z}}{-2i\bar{z} + 1}, \ \mathfrak{s}_4^\perp = \frac{(1-2i)\bar{z} + 2i}{-2i\bar{z} + 1 + 2i},$$

$$1 \to \Gamma \to PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle \to PGL_2(\mathbb{Z}[i]/(2)) \to 1,$$

$$[PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle : \Gamma] = 48, \ PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle = \Gamma \rtimes \text{Bin. Oct.}$$

Normal Form(s)

Given the commutation relations, there are two "natural" ways of writing words in the super-Apollonian generators.

We have *invert-first* normal form (push all $\mathfrak{s}_i^\perp$ as far left as possible)

$$\mathfrak{m}_1 \cdots \mathfrak{m}_n : \ \mathfrak{m}_k = \mathfrak{s}_i \text{ and } \mathfrak{m}_{k+1} = \mathfrak{s}_j^\perp \Rightarrow i = j,$$

and *swap-first* normal form (push all $\mathfrak{s}_i$ as far left as possible)

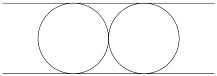
$$\mathfrak{m}_1 \cdots \mathfrak{m}_n : \ \mathfrak{m}_k = \mathfrak{s}_i^\perp \text{ and } \mathfrak{m}_{k+1} = \mathfrak{s}_j \Rightarrow i = j.$$

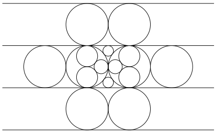
These provide two spanning trees for the Cayley graph of Γ with respect to the generators $\{\mathfrak{s}_i, \mathfrak{s}_i^\perp\}$.
For example:

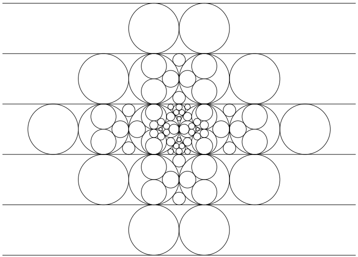
$$\mathfrak{s}_3^\perp \mathfrak{s}_1^\perp \mathfrak{s}_2 \mathfrak{s}_4^\perp \mathfrak{s}_1 \mathfrak{s}_3 \mathfrak{s}_2^\perp \mathfrak{s}_4 = \begin{cases} \mathfrak{s}_3^\perp \mathfrak{s}_1^\perp \mathfrak{s}_4^\perp \mathfrak{s}_1 \mathfrak{s}_2 \mathfrak{s}_2^\perp \mathfrak{s}_3 \mathfrak{s}_4 & \text{invert-first} \\ \mathfrak{s}_2 \mathfrak{s}_3^\perp \mathfrak{s}_1^\perp \mathfrak{s}_1 \mathfrak{s}_3 \mathfrak{s}_4^\perp \mathfrak{s}_4 \mathfrak{s}_2^\perp & \text{swap-first} \end{cases} .$$

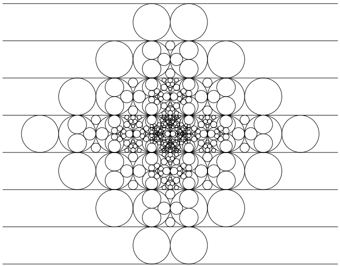
Apollonian Super-Packing

Starting with four mutually tangent circles (partitioning the plane into 4 triangles and 4 circles) we can iterate swaps and inversions, producing finer partitions of the plane into circles and triangles.



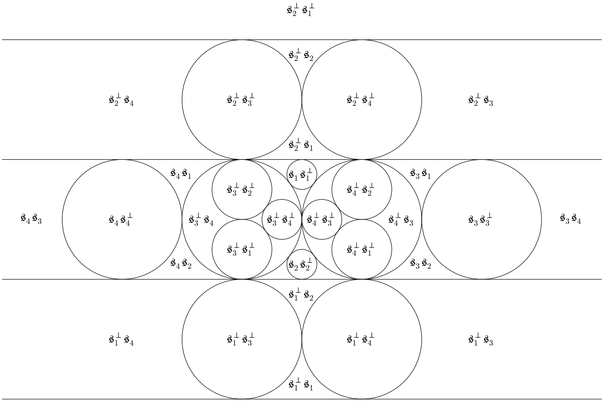






Apollonian Super-Packing (cont.)

The regions of the nth partition are labeled by the $9 \cdot 5^{n-1} - 1$ normal form words of length n in the super-Apollonian generators. Irrational z are uniquely coordinatized by infinite normal form words in the super-Apollonian generators $\{\mathfrak{s}_i, \mathfrak{s}_i^\perp\}$. The expansion of z in the super-Apollonian generators is produced by a dynamical system to be described below. [Regions labeled by words of length two shown.]



Apollonian Super-Packing (cont.)

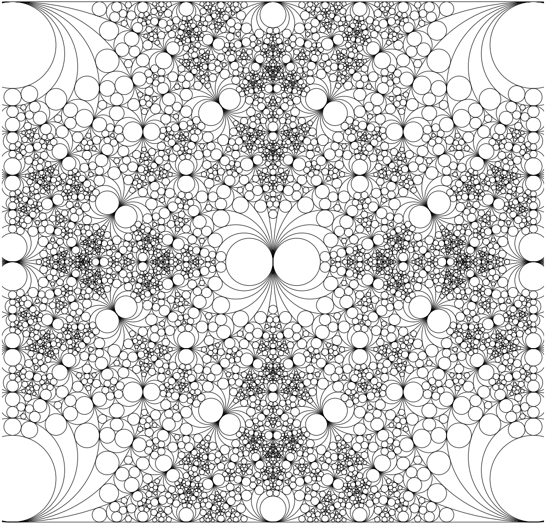
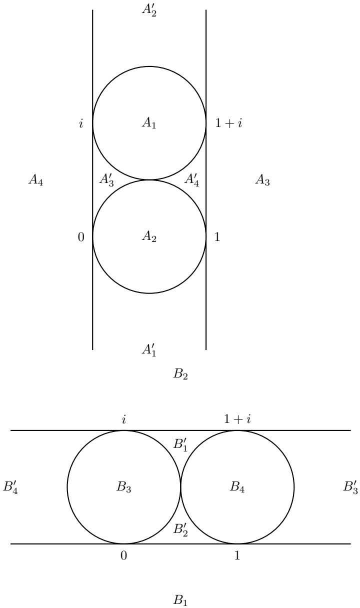


Figure: Portion of the sixth partition inside the unit square.

Dynamical System(s)



We define two dynamical systems defined with respect to the dual Descartes quadruples shown:

$$T_A(w) = \begin{cases} \mathfrak{s}_i w & w \in A_i, \\ \mathfrak{s}_i^{\perp} w & w \in A_i'. \end{cases},$$

$$T_B(z) = \begin{cases} \mathfrak{s}_i z & z \in B_i', \\ \mathfrak{s}_i^{\perp} z & z \in B_i, \end{cases}.$$

There are six neutral fixed points $\left\{0, 1, \infty, i, 1+i, \frac{1}{1-i}\right\}$ to which Gaussian rationals descend in finite time, depending on the parity of the numerator and denominator.

Dynamical Systems (cont.)

Recording the sequences $\mathfrak{m}_n$, $\mathfrak{n}_n$ defined by

$$T_A^n(w) = \mathfrak{m}_n(T_A^{n-1}(w)), \ T_B^n(z) = \mathfrak{m}_n(T_B^{n-1}(z))$$

produces infinite words in normal form. The initial segments $\mathfrak{m}_1 \cdots \mathfrak{m}_n$, $\mathfrak{n}_1 \cdots \mathfrak{n}_n$ label the region in the nth partition where w or z lies. We obtain 6 sequences of Gaussian rational approximations for each system such that

$$\begin{matrix} \lim\limits_{n\to\infty} \mathfrak{m}_1 \cdots \mathfrak{m}_n w_0 = w \\ \lim\limits_{n\to\infty} \mathfrak{n}_1 \cdots \mathfrak{n}_n z_0 = z \end{matrix} \quad w_0, z_0 \in \left\{0, 1, \infty, i, 1+i, \frac{1}{1-i}\right\},$$

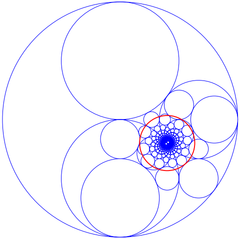
the vertices of the octahedra along the path indexed by the normal form word.

One measure of the quality of approximation is:

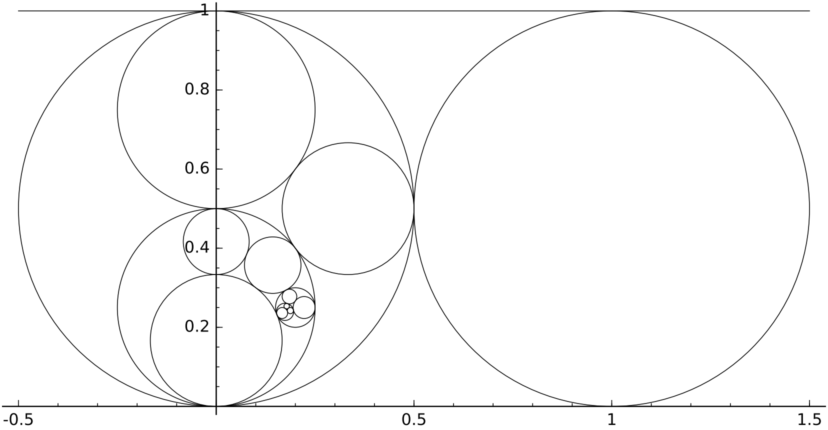
If $(p,q) = 1$ is such that

$$|z_0 - p/q| < \frac{C}{|q|^2}, \ C = \frac{1}{1+1/\sqrt{2}},$$

then p/q is a convergent to z_0. Moreover, C is the largest constant possible.



Example



$$z = 0.1761148094996705\ldots + i0.2463661645805464\ldots$$

$$\mathfrak{z} = \mathfrak{s}_3^\perp \mathfrak{s}_1^\perp \mathfrak{s}_2 \mathfrak{s}_2^\perp \mathfrak{s}_3^\perp \mathfrak{s}_3 \mathfrak{s}_1 \mathfrak{s}_3 \mathfrak{s}_3^\perp \mathfrak{s}_2^\perp \mathfrak{s}_1 \mathfrak{s}_4 \mathfrak{s}_1 \mathfrak{s}_4 \mathfrak{s}_4^\perp \mathfrak{s}_1 \mathfrak{s}_1^\perp \mathfrak{s}_2^\perp \mathfrak{s}_3 \mathfrak{s}_4 \ldots$$

Invertible Extension

The invertible extension $\widetilde{T}$ of T_B is defined on a space of geodesics $\mathcal{G} = \cup_i \left(\mathcal{B}_i \times B_i \cup \mathcal{B}'_i \times B'_i\right)$

$$\widetilde{T}(w,z) = \begin{cases} (\mathfrak{s}_i w, \mathfrak{s}_i z) & z \in B_i,\ \mathfrak{z} = \mathfrak{s}_i \dots \\ (\mathfrak{s}_i^{\perp} w, \mathfrak{s}_i^{\perp} z) & z \in B'_i,\ \mathfrak{z} = \mathfrak{s}_i^{\perp} \dots \end{cases}$$

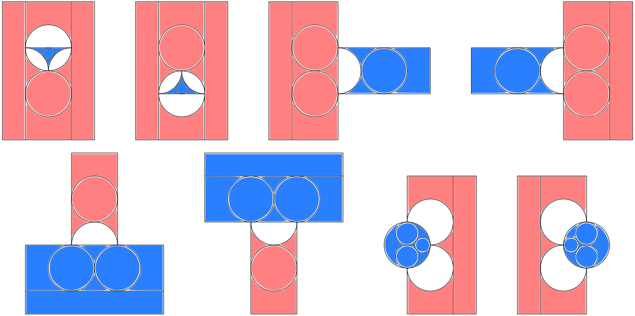
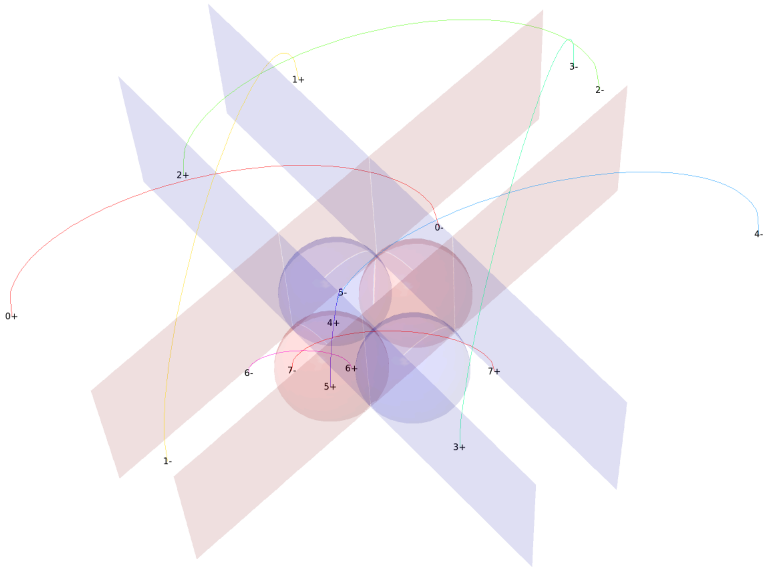


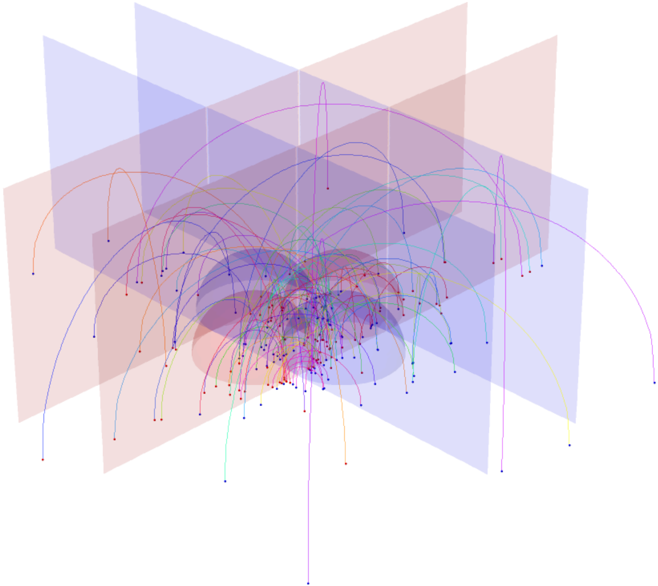
Figure: Regions $\mathcal{B}'_i \times B'_i$, $\mathcal{B}_i \times B_i$ with subdivisions.

Invertible Extension (cont.)



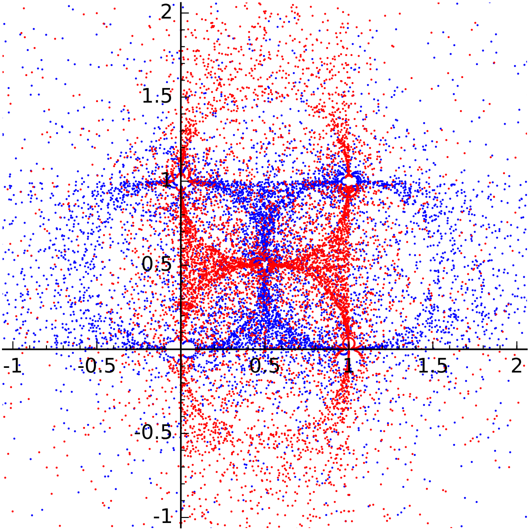
An example orbit, nth iteration labeled as $(n-, n+)$.

Invertible Extension (cont.)



A larger orbit (100 iterations on random input).

Invertible Extension (cont.)



The inverse $\widetilde{T}^{-1}$ extends T_A (in the same manner that $\widetilde{T}$ extends T_B) so that a geodesic (w, z) corresponds to a bi-infinite word $\mathfrak{w}^{-1}\mathfrak{z}$ in the super-Apollonian generators, with $\mathfrak{w}$ in swap-first normal form and $\mathfrak{z}$ in invert-first normal form. Hence working with one system automatically involves its dual.

[Pictured is part of 10,000 iterations of $\widetilde{T}$ on a random input (w, z),

$$z = 0.085432\ldots + i0.185957\ldots,$$
$$w = 0.491241\ldots + i0.343341\ldots.]$$

Invariant Measure

The measure $d\eta(w,z) = |z-w|^{-4}dudvdxdy$, $z = x+iy$, $w = u+iv$ is isometry-invariant on the space of geodesics in H^3. As $\widetilde{T}$ is a bijection defined piecewise by isometries, $\eta|_{\mathcal{G}}$ is $\widetilde{T}$-invariant. Pushing forward to the second coordinate gives a T_B-invariant measure μ_B on $P^1(\mathbb{C})$

$$d\mu_B(z) = f_B(z)dxdy = \begin{cases} dxdy\int_{\mathcal{B}_i} |z-w|^{-4}dudv, & z \in B_i \\ dxdy\int_{\mathcal{B}'_i} |z-w|^{-4}dudv, & z \in B'_i \end{cases} .$$

Invariant measure (cont.)

Computing these integrals gives:

(triangles) (circles)

$$f_B(x,y) = \begin{cases} \frac{\pi}{4\left(\frac{1}{4}-d_1^2\right)^2} & z \in B_1', \\ \frac{\pi}{4\left(\frac{1}{4}-d_2^2\right)^2} & z \in B_2', \\ \frac{\pi}{4(1-x)^2} & z \in B_3' \\ \frac{\pi}{4x^2} & z \in B_4' \end{cases}, \qquad f_B(x,y) = \begin{cases} H(x,y) & z \in B_1 \\ H(x,1-y) & z \in B_2 \\ G(x,y) & z \in B_3 \\ G(1-x,y) & z \in B_4 \end{cases},$$

where

$$d_1^2 = \left(x - \frac{1}{2}\right)^2 + (y-1)^2,\ d_2^2 = \left(x - \frac{1}{2}\right)^2 + y^2,$$

and

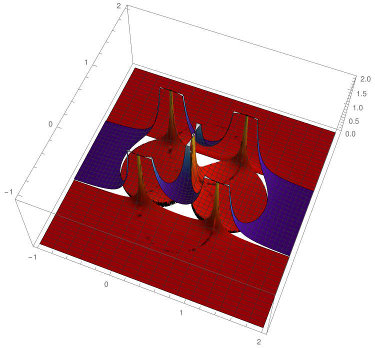
$$h(x,y) = \frac{\arctan(x/y)}{4x^2} - \frac{1}{4xy},$$

$$H(x,y) = h(x,y) + h(1-x,y) + h(x^2 - x + y^2, y),$$

$$G(x,y) = h(x, y^2 - y + x^2) + h(x^2 - x + y^2, y^2 - y + x^2) + h(x^2 - x + (1-y)^2, y^2 - y + x^2).$$

Invariant Measure (cont.)

The measure μ_B is **finite**, giving a measure of $\pi^2/4$ for each of the eight regions.



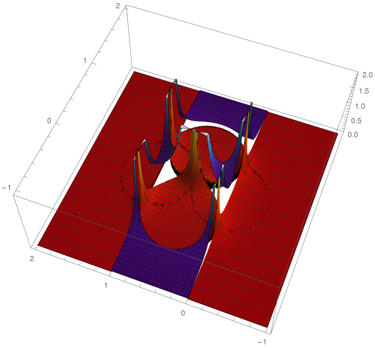


Figure: The density $f_B(z)$ shown from two angles (f_A is f_B rotated by $90°$).

The Work of A. L. Schmidt and H. Nakada

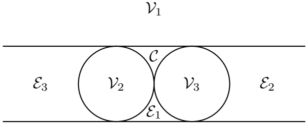


Figure: Model circle $\mathcal{I}$

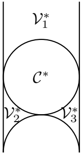


Figure: Model triangle $\mathcal{I}^*$

Asmus Schmidt (1975,1982) developed a pair of algorithms for continued fractions over $\mathbb{Q}(i)$ and studied their ergodic theory. Hitoshi Nakada (1988,1990) furthered their study. Let $X = \mathcal{I} \cup \mathcal{I}^*$ and define $T : X \to X$ by mapping the circular regions $\mathcal{V}_i$ and $\mathcal{C}^*$ onto $\mathcal{I}$ and the triangular regions $\mathcal{V}_i^*$ and $\mathcal{C}$ onto $\mathcal{I}^*$ with the inverses of the following Möbius transformations

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \; V_2 = SV_1S^{-1}, \; V_3 = S^2V_1S^{-2}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}, \; E_2 = SE_1S^{-1}, \; E_3 = S^2E_1S^{-2}$$

$$C = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}, \quad \text{and } S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

T is ergodic with respect to the measure assigning hyperbolic area to the triangle and the density $H(x, -y)$ on the upper half-plane.

The Work of A. L. Schmidt and H. Nakada (cont.)

If $T^n z = M_n^{-1} T^{n-1} z$ with $M_n \in \{V_i, E_i, C\}$, then following the orbits of $0, 1, \infty$ under the partial products $M_1 \cdot \dots \cdot M_n$ gives three sequences $p_\alpha^{(n)}/q_\alpha^{(n)}$, $\alpha = 0, 1, \infty$, of Gaussian rational approximations to z. Inducing to $X \setminus \cup_i (\mathcal{V}_i \cup \mathcal{V}_i^*)$ gives "faster" convergents $\hat{p}_\alpha^{(n)}/\hat{q}_\alpha^{(n)}$ and a sequence of exponents e_n (1 for E_i, C, and the return time k for V_i^k).

Some results for these convergents include

- [Schmidt] If $|z - p/q| < \frac{2}{(2+\sqrt{2})|q|^2}$ then $p/q = p_\alpha^{(n)}/q_\alpha^{(n)}$ for some n, α (noted earlier in the reflection group setting).
- [Nakada] If $|z - p/q| < \frac{1}{2|q|^2}$ then $p/q = \hat{p}_\alpha^{(n)}/\hat{q}_\alpha^{(n)}$ for some n, α.
- [Schmidt] The geometric and arithmetic means of the exponents exist almost everywhere

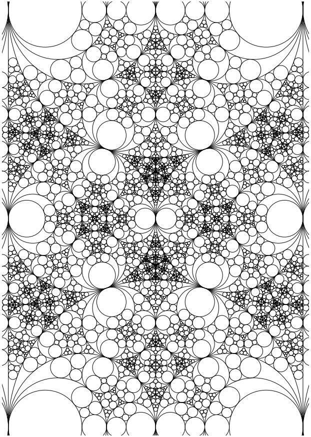
$$\lim_{n\to\infty} \left(\prod_{n=1}^{N} e_n \right)^{1/N} = 1.26\ldots, \quad \lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_n = 1.667\ldots.$$

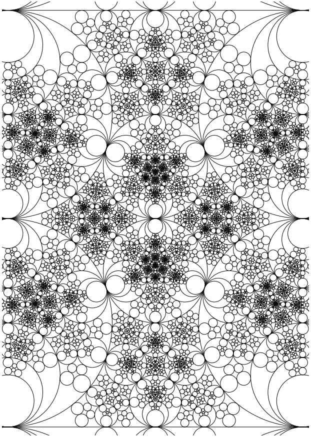
- [Nakada] The (almost everywhere) exponential rate of convergence is given by

$$\lim_{n\to\infty} \frac{1}{n} \log |z - p_\alpha^{(n)}/q_\alpha^{(n)}| = \frac{-2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Similar systems in $\mathbb{Q}(\sqrt{-2})$

There is a $GL_2(\mathbb{Z}[\sqrt{-2}]) \rtimes \langle \bar{z} \rangle$-invariant tessellation of H^3 by **right-angled** cuboctahedra and everything done above over $\mathbb{Q}(i)$ carries over to $\mathbb{Q}(\sqrt{-2})$ (dual pair of dynamical systems, rational approximations, invariant measure, 2-congruence, etc.).





Thanks!

Thank you for your attention.