Badly approximable numbers over imaginary quadratic fields

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The Euclidean algorthim [iterating $(a, b) \mapsto (b, a \bmod b)$]

$$\begin{aligned} a &= ba_0 + r_0,\ 0 \le r_0 < b \\ b &= r_0 a_1 + r_1,\ 0 \le r_1 < r_0 \\ r_0 &= r_1 a_2 + r_2,\ 0 \le r_2 < r_1 \\ &\dots, \end{aligned}$$

or written in matrices

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ r_0 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &\dots, \end{aligned}$$

expresses a rational number a/b as a finite continued fraction

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots \frac{1}{a_n}}}} =: [a_0; a_1, \dots, a_n].$$

Extending this to irrational numbers $\xi = \lfloor \xi \rfloor + \{\xi\} = a_0 + \xi_0$ gives a dynamical system

$$T : (0,1) \to (0,1),\ \xi_0 \mapsto \{1/\xi_0\}$$

and infinite sequences

$$\xi_{n+1} = \{1/\xi_n\} = T^{n+1}\xi_0,\ a_{n+1} = \lfloor 1/\xi_n \rfloor = \left\lfloor \frac{1}{T^n \xi_0} \right\rfloor$$

with

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0; a_1, a_2, \dots].$$

Stopping after n iterations gives rational approximations p_n/q_n to ξ, where

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

The branches of T^{-1} are all surjective and we have bijections

$$\mathbb{R} \setminus \mathbb{Q} \cong \mathbb{Z} \times \mathbb{N}^{\mathbb{N}}, \ \mathbb{Q} = \{[a_0; a_1, \ldots, a_n] : n \geq 0, \ a_n \neq 1 \text{ if } n \geq 1\}.$$

T is the left shift on these sequences, $T([0; a_1, a_2, \ldots]) = [0; a_2, a_3, \ldots]$.

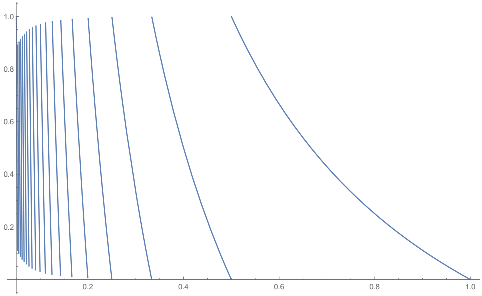


Figure: $T\xi = \{\frac{1}{\xi}\}$

For future use, we note that

$$q_n/q_{n-1} = [a_n; a_{n-1}, \ldots, a_1], \ q_n(q_n\xi - p_n) = \frac{(-1)^n}{q_{n+1}/q_n + \xi_{n+1}}.$$

The convergents p_n/q_n to ξ have the following properties.

Dirichlet bound

$$|\xi - p_n/q_n| \le 1/q_n^2$$

Best approximations

If $0 < q < q_n$ then $|q\xi - p| > |q_n\xi - p_n|$; i.e. the continued fraction convergents are the best rational approximations to ξ.

We say ξ is *badly approximable* if there exists $C' > 0$ such that

$$|\xi - p/q| \ge C'/q^2 \text{ for all } p, q \in \mathbb{Z},$$

i.e. the Dirichlet bound is tight (up to a multiplicative constant). This is an interesting class of real numbers (uncountable, measure zero, Hausdorff dimension 1, etc.) which is not completely understood.

One characterization of badly approximable numbers is the following.

The number $\xi \in \mathbb{R} \setminus \mathbb{Q}$ is badly approximable if and only its partial quotients are bounded.

If ξ is badly approximable, $|\xi - p/q| \geq C'/q^2$, then in particular

$$\frac{C'}{q_n^2} \leq |\xi - p_n/q_n| = \frac{1}{q_n^2([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1])} \leq \frac{1}{q_n^2 a_{n+1}},$$

$a_{n+1} \leq 1/C'$.

Conversely, if the partial quotients are bounded, $\sup_n \{a_n\} \leq M$, then for any p/q with $0 < q \leq q_n$

$$|\xi - p/q| \geq |\xi - p_n/q_n| = \frac{1}{q_n^2(q_{n+1}/q_n + \xi_{n+1})}$$

$$= \frac{1}{q_n^2([0; a_{n+2,\ldots}] + [a_{n+1}; a_n, \ldots, a_1])} \geq \frac{1}{q_n^2\,(a_{n+1} + 2)} \geq \frac{1}{q_n^2(M+2)},$$

using the fact that the convergents p_n/q_n are the best approximations.

Here is another characterization of badly approximable numbers.

Dani correspondence

The number ξ is badly approximable if and only if the trajectory

$$\Omega_\xi = \left\{ SL_2(\mathbb{Z}) \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$$

is bounded (precompact).

We can think of Ω_ξ as a geodesic ray on the modular surface, "aimed" at ξ [$PSL_2(\mathbb{R})$ can be identified with the unit tangent bundle $T^1(\mathbb{H}^2)$], or as a one-parameter family of two-dimensional unimodular lattices.

The proof invokes Mahler's criterion (slight overkill in dimension 2):

Mahler's criterion

$X \subseteq SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$ is precompact iff the lengths of all vectors in the corresponding unimodular lattices $\Lambda_x, x \in X$ are uniformly bounded below.

Suppose ξ is badly approximable with $|q(q\xi + p)| \geq C'$ for all p, q. If

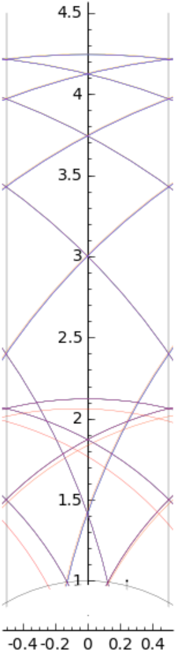
$$\left\| (q\ p) \left(\begin{array}{cc} 1 & \xi \\ 0 & 1 \end{array} \right) \left(\begin{array}{cc} e^{-t} & 0 \\ 0 & e^{t} \end{array} \right) \right\|_{\infty} = \|(e^{-t}q, e^{t}(q\xi + p))\|_{\infty} < \sqrt{C'},$$

then taking the product of the coordinates gives a contradiction. Hence Ω_ξ is precompact by Mahler's criterion.

Conversely, if ξ is not badly approximable, there exist sequences a_n, b_n such that $|b_n(b_n\xi + a_n)| \leq 1/n^2$. If t_n is such that $e^{-t_n}|b_n| \leq 1/n$, then

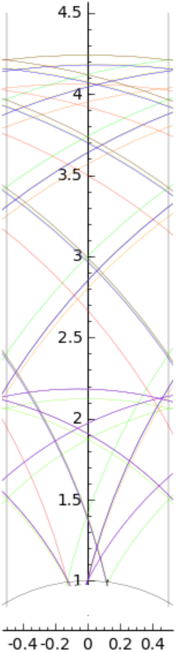
$$\|(e^{-t_n}b_n, e^{t_n}(b_n\xi + a_n))\|_{\infty} \leq 1/n,$$

and the trajectory Ω_ξ is not bounded (Mahler's criterion again).



Left is the trajectory aimed at $\xi = 3\sqrt{2} - 4 = [0; \overline{4, 8}]$. The trajectory is bounded, asymptotic to the closed geodesic joining the conjugate points ξ, $\bar{\xi}$.

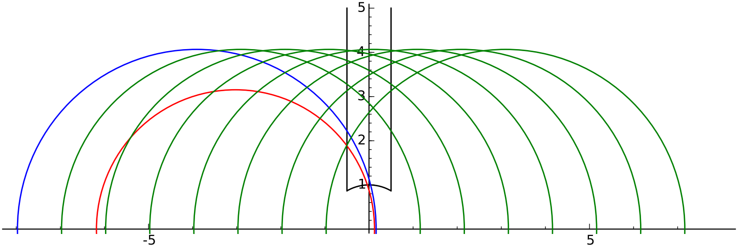
Right is the trajectory aimed at the transcendental $\xi = [0; 4, 8, 8, 4, 8, 4, 4, 8, \dots]$ (digits given by the Thue-Morse sequence on $\{4, 8\}$).



The last two characterizations are seen to be equivalent by considering the orbit of the geodesic (∞, ξ_0) under the invertible extension of the Gauss map

$$\widetilde{T}^n(\infty, \xi_0) = (-q_n/q_{n-1}, \xi_n) \in (-a_n - 1, -a_n) \times (0, 1).$$

The diameter of the geodesics above are bounded between a_n and $a_n + 2$. The partial quotients measure the height of the "excursions into the cusp" of the geodesic trajectory.



$(-q_{n-1}/q_{n-2}, \xi_{n-1})$, $(q_n/q_{n-1} - a_n, -\xi_n - a_n)$, translates
One excursion into the cusp, $a_n = 7$.

The obvious (and conjecturally only) algebraic numbers badly approximable over $\mathbb{Q}$ are quadratic irrationals. Here is an algebraic proof.

Let

$$Q(x,y) = ax^2 + bxy + cy^2 = (x\ y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

$$a,b,c \in \mathbb{Z},\ \Delta(Q) = ac - b^2/4 < 0,$$

be an indefinite integral binary quadratic form, and $Z(Q)$ its zero set (i.e. two points in $P^1(\mathbb{R})$). $GL_2(\mathbb{Z})$ acts by change of variable on Q, by Möbius transformations on $Z(Q)$, and the map $Q \mapsto Z(Q)$ is $GL_2(\mathbb{Z})$-equivariant:

$$Z((g^{-1})^t Q g^{-1}) = g \cdot Z(Q).$$

For $\xi \in \mathbb{R}$, $\xi = [a_0; a_1, \ldots]$, let

$$g_n^{-1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

so that $T^n(\xi_0) = 1/g_n\xi$. If $Q(\xi, 1) = 0$ for some indefinite integral binary quadratic form, let $Q_n = {}^{g_n}Q$, and note that $Q_n(1, \xi_n) = 0$.

One can easily bound the coefficients of the collection $\{Q_n\}_n$ to show the following.

The orbit $\{Q_n\}_n$ is finite (and eventually periodic).

This shows that the sequence ξ_n is finite (and evetually periodic), so that the a_n are bounded (and eventually periodic). Hence ξ is badly approximable.

We can also see that quadratic irrationals are badly approximable from a geometric point of view using the Dani correspondence.

The closed (compact/periodic) geodesics on the modular surface are the projections of the geodesics in $\mathbb{H}^2$ joining conjugate real quadratic irrationals (upper half-plane model). [Consider the fixed points $z = \frac{az+b}{cz+d}$ of a hyperbolic element of $SL_2(\mathbb{Z})$.]

If $Q(\xi, 1) = 0$ for an integral form, then the geodesic trajectory Ω_ξ is asymptotic to the closed geodesic joining ξ, $\bar{\xi}$. Therefore Ω_ξ is bounded and ξ is badly approximable.

A long-standing open question is the following.

If ξ is algebraic and badly approximable, is ξ quadratic?

As we saw above, there is an obvious geometric reason that quadratics are badly approximable. However there are uncountably many transcendental real numbers whose partial quotients and associated geodesic trajectories are bounded.

algebraic + badly approximable $\Rightarrow$ compactness?

[AFAIK, there are are no known examples of algebraic numbers with *unbounded* partial quotients, although numerical evidence seems to show that the partial quotients of higher degree algebraic numbers follow the Gauss-Kuzmin statistics.]

We want to generalize the above to give examples of badly approximable numbers over any imaginary quadratic field K, using the nearest integer continued fractions of Hurwitz if $\mathcal{O}_K$ is Euclidean,

$$K = \mathbb{Q}(\sqrt{d}),\ d = -1, -2, -3, -7, -11,$$

and appealing to geodesic trajectories when $\mathcal{O}_K$ is not Euclidean.

[What follows generalizes/uses/overlaps work of D. Hensely, R. Lakein, W. Bosma and D. Gruenewald, and S. G. Dani.]

Let $K = \mathbb{Q}(\sqrt{d})$, $d = -1, -2, -3, -7, -11$, an imaginary quadratic field whose maximal order is Euclidean, and let be V the complex numbers closer to zero than to any other point of $\mathcal{O}_K$ along with a choice of half the boundary. Any $z \in \mathbb{C}$ can be uniquely written as

$$z = a_0 + z_0,\ a_0 =: \lceil z \rfloor \in \mathcal{O}_K,\ z_0 =: \{z\} \in V.$$

Define $T : V \to V$ by $Tz = \{1/z\}$. For $z \in \mathbb{C}$, define

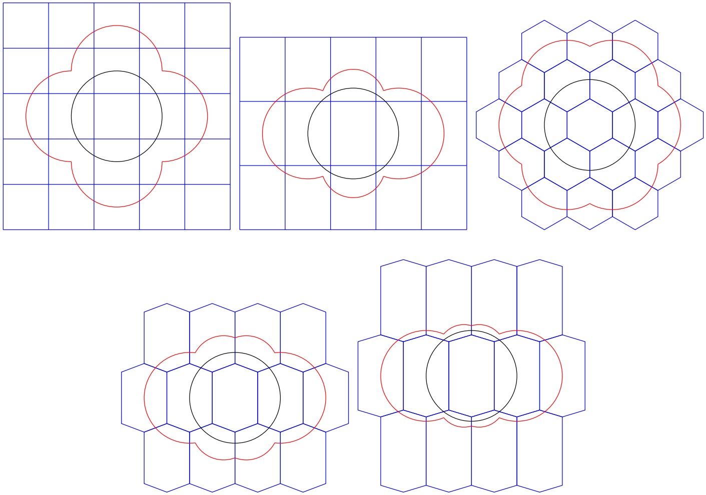
$$z_n = T^n z_0 = \{1/z_{n-1}\},\ a_n = \lceil 1/z_{n-1} \rfloor,$$

expressing z as a continued fraction

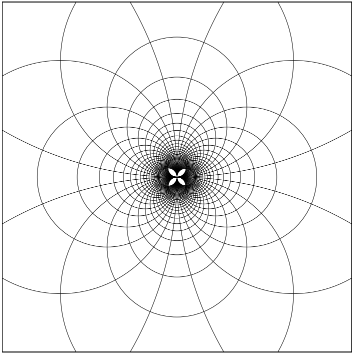
$$z = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

with convergents $p_n/q_n = a_0 + \frac{1}{a_1 + \frac{1}{\ldots + \frac{1}{a_n}}} \in K$

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$



V and its translates in blue, unit circle in black, and $\partial(V^{-1})$ in red.
[Also a "proof by picture" that $\mathcal{O}_K$ is norm Euclidean.]



Partition of V induced by one iteration of T.

The Hurwitz continued fractions aren't as nice as simple continued fractions, for various reasons.

One reason is that the branches of the inverse aren't surjective near the boundary of V, making the sequence space $[a_0; a_1, a_2, \ldots]$ hard to describe (e.g. there are arbitrarily long "forbidden sequences" of digits in nearest integer continued fractions).

Another is that they don't always give the best rational approximations relative to the norm of the denominator.

While the nearest integer convergents p_n/q_n to $z \in \mathbb{C}$ are not necessarily the best rational approximations to z, they aren't so bad.

Dirichlet bound

There exists $C > 0$ such that the convergents p_n/q_n to any $z \in \mathbb{C}$ satisfy

$$|z - p_n/q_n| \leq C/|q_n|^2.$$

OK approximations

There exists $\alpha > 0$ such that for any $z \in \mathbb{C}$, $p, q \in \mathcal{O}_K$, $|q| \leq |q_n|$,

$$\alpha|qz - p| \geq |q_n z - p_n|.$$

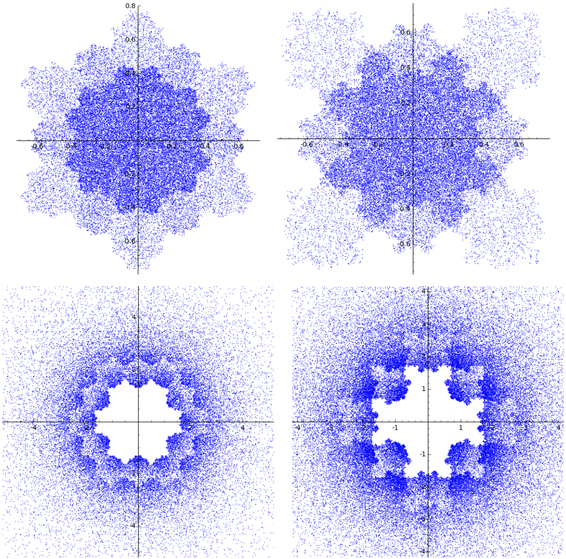
The second statement is essentially due to R. Lakein, who found

$$\sup_{z,n} |q_n(q_n z - p_n)|$$

for the algorithms considered here. It can also be found implicitly in the work of D. Hensley and explicitly S. G. Dani over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively.

Monotonicity of q_n

The proofs of the above require monotonicity of the convergent denominators, $|q_{n-1}| < |q_n|$, somewhat annoying to establish, and related to the difficulty of describing the natural extension of T. Shown are $\frac{q_{n-1}}{q_n}$, $\frac{q_n}{q_{n-1}}$ for $1 \leq n \leq 10$ and 5000 random z, over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$.



While the nearest integer convergents are not the best approximations available, they are “good enough” to detect badly approximable numbers.

The number $z \in \mathbb{C} \setminus K$ is badly approximable over K if and only if its partial quotients are bounded.

The proof follows from the approximation properties described earlier, with the “OK approximations” allowing us to say that a number is badly approximable if and only if it is badly approximable by its convergents. [This statement for $\mathbb{Q}(\sqrt{-3})$ can also be found in a recent preprint of S. G. Dani.]

The zero set $Z(H)$ of the indefinite binary Hermitian form

$$H(z,w) = (\bar{z}\ \bar{w}) \begin{pmatrix} A & -B \\ -\overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = A|z|^2 - B\bar{z}w - \overline{B}z\bar{w} + C|w|^2,$$

$$A, C \in \mathbb{R}, B \in \mathbb{C},\ \Delta(H) := \det(H) < 0,$$

is a circle in $P^1(\mathbb{C})$; e.g. if $A \neq 0$ then

$$Z(H) \cap \mathbb{C}_z = \{z : |z - B/A|^2 = -\Delta/A^2\}.$$

$GL_2(\mathbb{C})$ acts on a form H by change of variable and on the circle $Z(H)$ by the usual Möbius action, and the map $H \to Z(H)$ is $GL_2(\mathbb{C})$-equivariant:

$$Z((g^{-1})^\dagger H g^{-1}) = g \cdot Z(H),\ g \in GL_2(\mathbb{C}).$$

A form/circle is *rational* if $A, B, C \in K$ and *integral* if $A, B, C \in \mathcal{O}_K$. We can restrict the actions above to $GL_2(\mathcal{O}_K)$ and integral forms.

For $z \in \mathbb{C}$ with $z = [a_0; a_1, \ldots]$, define

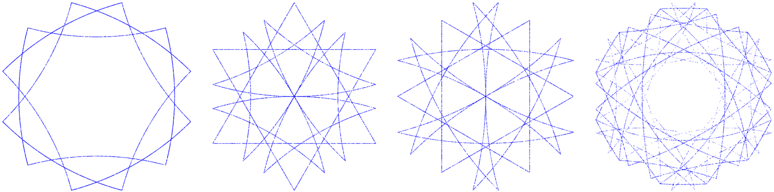
$$g_n^{-1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

so that $T^n(z_0) = 1/g_n z$. If $H(z, 1) = 0$ for an indefinite integral binary Hermitian form, let $H_n = {}^{g_n}H$, so that $H_n(1, z_n) = 0$.

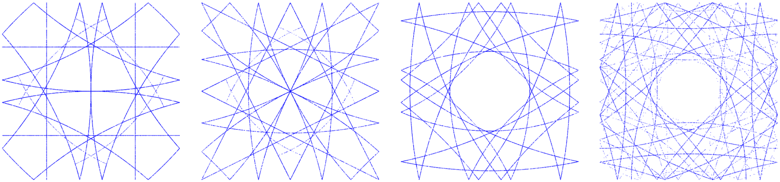
The orbit $\{H_n\}_n$ is finite.

This is analogous to the fact that real irrational quadratic numbers have eventually periodic simple continued fraction expansions as discussed earlier.

However, the sequences H_n and z_n aren't periodic (unless z is a quadratic irrational); there is room to move around on the tagged circles $(z_n, Z(H_n))$. [This generalizes work of Wieb Bosma and David Gruenewald over $\mathbb{Q}(\sqrt{-1})$.]



Example orbits $\{z_n\}_n$, $0 \leq n \leq 10,000$ for zeros z of various integral forms over $\mathbb{Q}(\sqrt{-3})$.



Example orbits $\{z_n\}_n$, $0 \leq n \leq 20,000$ for zeros z of various integral forms over $\mathbb{Q}(\sqrt{-1})$.

[For the rest of the talk K is *any* imaginary quadratic field.]
We noted earlier that the zero sets of indefinite quadratic forms determined geodesics on the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}^2$. In a similar fashion, the zero sets of indefinite Hermitian forms determine geodesic surfaces in the *Bianchi orbifolds* $SL_2(\mathcal{O}_K)\backslash\mathbb{H}^3$.

A circle $Z(H)$ in the plane determines a geodesic plane (hemisphere) $S(H)$ in $\mathbb{H}^3$ (upper half-space model). In the quotient $\pi : \mathbb{H}^3 \to SL_2(\mathcal{O}_K)\backslash\mathbb{H}^3$ we get some geodesic surface $\pi(S(H))$.

If H is an anisotropic rational form, then $\pi(S(H))$ is compact.

[H is *anisotropic* if $H(z, w) \neq 0$ for $[z : w] \in P^1(K)$. This is equivalent to the condition

$$-\Delta(H) \notin N^K_{\mathbb{Q}}(K),$$

i.e. the square of the radius of the rational circle is not a norm.]

For any imaginary quadratic field K, the complex numbers badly approximable over K can be characterized dynamically.

Dani correspondence

The number z is badly approximable if and only if the trajectory

$$\Omega_z = \left\{ SL_2(\mathcal{O}_K) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathcal{O}_K) \backslash SL_2(\mathbb{C})$$

is bounded (precompact).

We can think of Ω_z as a framed geodesic trajectory aimed at z [$PSL_2(\mathbb{C})$ can be identified with the oriented orthonormal frame bundle of $\mathbb{H}^3$] or as a one parameter family of two-dimensional unimodular $\mathcal{O}_K$-lattices.

The proof is the same as that given earlier, restricting scalars from K to $\mathbb{Q}$ and using Mahler's criterion in dimension four.

Synthesizing the above, we can produce infinitely many circles which consist entirely of badly approximable numbers (over any imaginary quadratic field).

If $z \in \mathbb{C}$ satisies $H(z,1) = 0$ for some anisotropic integral form H, then z is badly approximable.

- If K is Euclidean, then $\{Z(H_n)\}_n$ is a finite collection of circles bounded away from zero/infinity so that the partial quotients of z are bounded (and all approximation constants are effective).
- For general K, the trajectory Ω_z is asymptotic to the compact geodesic surface $\pi(S(H))$, and is therefore bounded.

[The collection of badly approximable points produced is uncountable of measure zero, dense in the plane, and of Hausdorff dimension 1. The collection of all numbers badly approximable over K has Hausdorff dimension 2.]

On these anisotropic circles, there are many examples of algebraic numbers badly approximable over K.

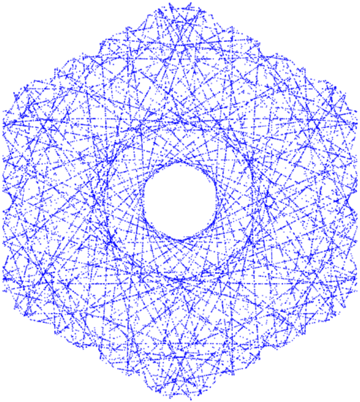
For any real algebraic number $u \in [-2, 2]$, any $0 < n \in \mathbb{Q} \setminus N_{\mathbb{Q}}^{K}(K)$, and any $t \in K$, the number

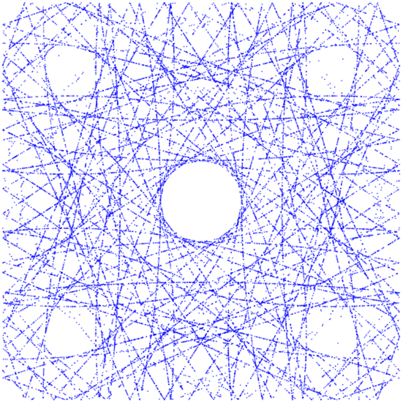
$$z = t + \sqrt{n} \cdot \frac{u \pm \sqrt{u^2 - 4}}{2}$$

is badly approximable. Moreover, this parameterizes all of the algebraic numbers badly approximable over K coming from rational circles.

[Examples of algebraic numbers with bounded partial quotients over $\mathbb{Q}(\sqrt{-1})$ were given by Bosma and Gruenewald generalizing examples of Hensley.]

For instance, 30,000 iterates of T on the quadratically scaled root of unity $z = \sqrt{23}e^{2\pi i/5}$ over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$ $(23 \notin N_{\mathbb{Q}}^{K})$:





The fact that zeros of anisotropic indefinite integral binary Hermitian forms are badly approximable also has a trivial proof à la Liouville. If $H(z,1) = 0$ and $|z - p/q| \leq 1$, then by the mean value theorem

$$|H(p/q,1)| = |H(z,1) - H(p/q,1)| \leq c_1 |z - p/q|$$

for some $c_1 > 0$. Because H is anisotropic and integral,

$$|q|^2 |H(p/q,1)| \geq c_2$$

where $c_2 = \min\{|a| : 0 \neq a \in \mathcal{O}_K\}$. Hence

$$|z - p/q| \geq \frac{c_1/c_2}{|q|^2}.$$

An extension of the folklore conjecture/open question posed earlier is the following.

If z is algebraic and badly approximable over K, does z lie on a rational circle?

Or, once again,

algebraic + badly approximable $\Rightarrow$ compactness?

☺ Thank you for your attention ☺

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