Applications of hyperbolic geometry to continued fractions and Diophantine approximation

by

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Applications of hyperbolic geometry to continued fractions and Diophantine approximation

Thesis directed by Prof. Katherine E. Stange

This dissertation explores relations between hyperbolic geometry and Diophantine approximation, with an emphasis on continued fractions over the Euclidean imaginary quadratic fields, $\mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$, and explicit examples of badly approximable numbers/vectors with an obvious geometric interpretation.

The first three chapters are mostly expository. Chapter 1 briefly recalls the necessary hyperbolic geometry and a geometric discussion of binary quadratic and Hermitian forms. Chapter 2 briefly recalls the relation between badly approximable systems of linear forms and bounded trajectories in the space of unimodular lattices (the *Dani correspondence*). Chapter 3 is a survey of continued fractions from the point of view of hyperbolic geometry and homogeneous dynamics. The chapter discusses simple continued fractions, nearest integer continued fractions over the Euclidean imaginary quadratic fields, and includes a summary of A. L. Schmidt's continued fractions over $\mathbb{Q}(\sqrt{-1})$.

Chapters 4 and 5 contain the bulk of the original research. Chapter 4 discusses a class of dynamical systems on the complex plane associated to polyhedra whose faces are two-colorable (i.e. edge-adjacent faces do not share a color). To any such polyhedron, one can associated a right-angled hyperbolic Coxeter group generated by reflections in the faces of a (combinatorially equivalent) right-angled ideal polyhedron in hyperbolic 3-space. After some generalities, we discuss a simpler system, billiards in the ideal hyperbolic triangle. We then discuss continued fractions over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ coming from the regular ideal right-angled octahedron and cubeoctahedron.

Chapter 5 gives explicit examples of numbers/vectors in $\mathbb{R}^r \times \mathbb{C}^s$ that are badly approximable over number fields F of signature (r, s) with respect to the diagonal embedding. One should think of these examples as generalizations of real quadratic irrationalities, which we discuss first as our

prototype. The examples are the zeros of (totally indefinite anisotropic F-rational) binary quadratic and Hermitian forms (the Hermitian case arises when F is CM). Such forms can be interpreted as compact totally geodesic subspaces in the relevant locally symmetric spaces $SL_2(\mathcal{O}_F)\backslash SL_2(F\otimes\mathbb{R})/SO_2(\mathbb{R})^r\times SU_2(\mathbb{C})^s$. We discuss these examples from a few different angles: simple arguments stemming from Liouville's theorem on rational approximation to algebraic numbers, arguments using continued fractions (of the sorts considered in chapters 3 and 4) when they are available, and appealing to the Dani correspondence in the general case. Perhaps of special note are examples of badly approximable *algebraic* numbers and vectors, as noted in 5.10.

Chapter 6 considers approximation in $\mathbb{R}^n$ (in the boundary $\partial\mathcal{H}^n$ of hyperbolic $n-$space) over "weakly Euclidean" orders in definite Clifford algebras. This includes a discussion of the relevant background on the "SL_2" model of hyperbolic isometries (with coefficients in a Clifford algbra) and a discription of the continued fraction algorithm. Some exploration in the case $\mathbb{Z}^3\subseteq\mathbb{R}^3$ is included, along with proofs that zeros of anisotropic rational Hermitian forms are "badly approximable," and that the partial quotients of such zeros are bounded (conditional on increasing convergent denominators).

Chapter 7 considers simultaneous approximation in $\mathbb{R}^r\times\mathbb{C}^s$ as a subset of the boundary of $(\mathcal{H}^2)^r\times(\mathcal{H}^3)^s$ over a diagonally embedding number field of signature (r,s). A continued fraction algorithm is proposed for norm-Euclidean number fields, but not even convergence is established. Some exploration and experimentation over the norm-Euclidean field $\mathbb{Q}(\sqrt{2})$ is included.

Finally, chapter 8 includes some miscellaneous results related to the discrete Markoff spectrum. First, some identities for sums over Markoff numbers are proven (although they are closely related to Mcshane's identity). Secondly, transcendence of certain limits of roots of Markoff forms is established (a simple corollary to [AB05]). These transcendental numbers are badly approximable with only ones and twos in their continued fraction expansion, and can be written as infinite sums of ratios of Markoff numbers indexed by a path in the tree associated to solutions of the Markoff equation $x^2+y^2+z^2=3xyz$. The geometry in this chapter can all be associated to a once-punctured torus (with complete hyperbolic metric), going back to the observation of H. Cohn [Coh55] that the Markoff equation is a special case of Fricke's trace identity

$$tr(A)^2 + tr(B)^2 + tr(AB)^2 = tr(A)tr(B)tr(AB) + tr(ABA^{-1}B^{-1}) + 2, \;\; A, B \in SL_2$$

in the case that A and B are hyperbolic with parabolic commutator of trace -2 (in particular for the torus associated to the commutator subgroup of $SL_2(\mathbb{Z})$).

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Chapter 1

Some hyperbolic geometry

1.1 Introduction

In this chapter we recall some notions from hyperbolic geometry and the parameterization of pertinent geometric objects by algebraic ones. We are primarily interested in the Lie groups $G = SL_2(\mathbb{R}^r \times \mathbb{C}^s)$ and the arithmetic lattices (finite covolume discrete subgroups) $\Gamma = SL_2(\mathcal{O}_F)$ therein (where $\mathcal{O}_F$ is the ring of integers in the number field F of signature (r, s)). The associated symmetric spaces G/K, $K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s$, are products of hyperbolic two- and three-spaces, and $G/\{\pm 1\}^{r+s}$ is the group of orientation preserving isometries, acting on the left.

Hyperbolic n-space, $\mathcal{H}^n$, is the unique complete simply connected Riemannian n-manifold of constant negative sectional curvature -1 ([Lee97] Theorem 11.12), one of the prototypical model geometries (along with flat Euclidean space, curvature 0, and the round unit sphere, curvature $+1$). There are various models of hyperbolic space, summarized below.

- (hyperboloid) This model is (one sheet of, or the quotient by ± 1 of) the hypersurface in flat $\mathbb{R}^{n+1}$ defined by $x_0^2 - \sum_{i=1}^n x_i^2 = 1$ with the metric $ds^2 = -dx_0^2 + \sum_{i=1}^n dx_i^2$ (which is positive definite when restricted to the hyperboloid). Its orientation preserving isometry group is $O_{\mathbb{R}}(n, 1)^\circ$, the connected component of the identity of the real orthogonal group preserving the quadratic form in the definition of the metric above. The totally geodesic subspaces are the intersections of linear subspaces of $\mathbb{R}^{n+1}$ with the hyperboloid.

- (conformal ball) Projecting from the point $(-1, 0, \ldots, 0)$ to the hyperplane $x_0 = 0$ in the hyperboloid model gives the ball $x_0 = 0$, $\sum_{i=1}^n x_i < 1$, with metric $ds^2 = \frac{4\sum_i dx_i^2}{(1-\sum_i x_i^2)^2}$. Totally geodesic subspaces are hemispheres orthogonal to the boundary of the ball. The model is conformal, i.e. the Euclidean angles and hyperbolic angles agree. The isometry group is generated by inversions in the codimension 1 hemispheres orthogonal to the boundary (i.e. inversive geometry in $\mathbb{R}^n$ preserving the unit ball).

- (upper half-space) For this model, we take $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ with the metric $ds^2 = \frac{\sum_i dx_i^2}{x_n^2}$. Totally geodesic subspaces are hemispheres and affine subspaces orthogonal to the boundary $x_n = 0$. The isometry group is generated by inversions in the hemispheres and reflections in the affine subspaces of codimension 1 orthogonal to the boundary (i.e. inversive geometry of $\mathbb{R}^n$ preserving the upper half-space). Inversion (in $\mathbb{R}^n$) in the sphere of radius $\sqrt{2}$ centered at $(0, \ldots, 0, 1)$ exchanges the conformal ball and upper half-space models.

- (projective ball) Projecting from the origin in $\mathbb{R}^{n+1}$ to the plane $x_0 = 1$ in the hyperboloid model gives the projective ball model, $x_0 = 1$, $\sum_{i=1}^n x_i^2 < 1$. Totally geodesic subspaces are (Euclidean) affine subspaces in the ball. This model is not conformal (angles are not preserved).

We will work primarily (exclusively?) with the upper half-space model since we are interested in specific discrete subgroups of the isometry group of this model.

As a prototype for our investigations (cf. chapter 3), we consider the approximation of a real number by rational numbers. Homogenizing the problem, we want to work with $P^1(\mathbb{Q}) = P^1(\mathbb{Z})$ inside of $P^1(\mathbb{R})$. $P^1(\mathbb{Q})$ is the orbit of the point $\infty = [1 : 0]$ under the action of the lattice $\Gamma = SL_2(\mathbb{Z}) \subseteq G = SL_2(\mathbb{R})$. The Euclidean algorithm on $\mathbb{Z}$ leads to continued fractions and various questions in Diophantine approximation can be restated in terms of the geodesic flow on the modular surface $\Gamma\backslash G/K$, $K = SO_2(\mathbb{R})$.

We leave discussion of higher dimensional hyperbolic spaces and their "SL_2" isometry groups to chapter 6. Some references for hyperbolic geometry, discrete subgroups of isometries, and their arithmetic include [Rat06], [Bea95], [EGM98], and [MR03].

1.2 Upper half-space model in dimensions two and three

In dimension two, we identify $\mathcal{H}^2$ with a subset of the complex plane and the ideal boundary with the real projective line

$$\mathcal{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}, \ \partial\mathcal{H}^2 = P^1(\mathbb{R}).$$

With these coordinates, the metric is given by $ds^2 = \frac{dx^2+dy^2}{y^2}$. Geodesics are vertical half-lines or semi-circles orthogonal to the real line.

In this model we have

$$\mathrm{Isom}(\mathcal{H}^2) = PGL_2(\mathbb{R}) = GL_2(\mathbb{R})/\mathbb{R}^\times,$$

acting as fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & ad - bc > 0, \\ \frac{a\bar{z}+b}{c\bar{z}+d} & ad - bc < 0. \end{cases}$$

This action extends continuously to the ideal boundary.

In a similar fashion, we identify $\mathcal{H}^3$ with a subset of the Hamiltonians $\mathbb{H}$ and the ideal boundary with the complex projective line

$$\mathcal{H}^2 = \{\zeta = z + jt \in \mathbb{H} : z \in \mathbb{C}, t > 0\}, \ \partial\mathcal{H}^3 = P^1(\mathbb{C}).$$

With these coordinates, the metric is given by $ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}$ (with $z = x + iy$). Geodesics are vertical half-lines or semi-circles orthogonal to the complex plane.

In this model, the orientation preserving isometries are

$$\mathrm{Isom}^+(\mathcal{H}^3) = PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/\mathbb{C}^\times,$$

acting as fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \zeta = (a\zeta + b)(c\zeta + d)^{-1}$$

Once again this action extends continuously to the ideal boundary (or vice versa, the so-called Poincaré extension).

One can continue this process, identifying a group of two-by-two matrices over definite Clifford algebras with the orientation preserving isometries in the upper half-space mode. This generalization will be discussed in chapter 6.

1.3 Binary quadratic and Hermitian forms

The points of $\mathcal{H}^2$ and $\mathcal{H}^3$ can be identified with zero sets of definite binary quadratic and Hermitian forms, for instance the map

$$G \to G/K,\ g \mapsto g^\dagger g,$$

where $G = SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$, $K = SO_2(\mathbb{R})$ or $SU_2(\mathbb{C})$, and $\dagger$ is conjugate-transpose, identifies the symmetric space $G/K \cong \mathcal{H}^2$ or $\mathcal{H}^3$ with positive definite binary quadratic or Hermitian forms of determinant 1. Conversely, given a positive definite binary quadratic or Hermitian form

$$F(x,y) = ax^2 + bxy + cy^2 \text{ or } H(z,w) = Az\bar{z} + \bar{z}Bw + \bar{w}\overline{B}z + Cw\bar{w},$$

$$a,b,c \in \mathbb{R},\ ac - b^2/4 > 0,\ A,C \in \mathbb{R},\ B \in \mathbb{C},\ \Delta := \det(H) = AC - B\overline{B} > 0,$$

we can associate the following point of $\mathcal{H}^2$ or $\mathcal{H}^3$

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \in \mathcal{H}^2,\ \zeta = \frac{-B + j\sqrt{\Delta}}{A} \in \mathcal{H}^3$$

giving an SL_2-equivariant bijection between definite forms (modulo real scalar multiplication) and $\mathcal{H}^2$ or $\mathcal{H}^3$

$$Z(F^g) = g^{-1} \cdot Z(F) \text{ or } Z(H^g) = g^{-1} Z(H),$$

where the right action on forms is given by change of variable

$$F^g = g^t F g = \begin{pmatrix} \alpha^2 a + \alpha\gamma b + \gamma^2 c & \alpha\beta a + \frac{\alpha\delta+\beta\gamma}{2} b + \gamma\delta c \\ \alpha\beta a + \frac{\alpha\delta+\beta\gamma}{2} b + \gamma\delta c & \beta^2 a + \beta\delta b + \delta^2 c \end{pmatrix},$$

$$H^g = g^\dagger H g = \begin{pmatrix} |\alpha|^2 a + \alpha\bar{\gamma}\bar{b} + \bar{\alpha}\gamma b + |\gamma|^2 c & \bar{\alpha}\beta a + \beta\bar{\gamma}\bar{b} + \delta\bar{\alpha} b + \bar{\gamma}\delta c \\ \alpha\bar{\beta} a + \alpha\bar{\delta}\bar{b} + \bar{\beta}\gamma b + \gamma\bar{\delta} c & |\beta|^2 a + \beta\bar{\delta}\bar{b} + \bar{\beta}\delta b + |\delta|^2 c \end{pmatrix},$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and the left action on the the zero set is the restriction of the isometric action to the boundary (linear fractional transformation),

$$g \cdot [z : w] = [az + bw : cz + dw].$$

An unoriented geodesic in $\mathcal{H}^2$ or $\mathcal{H}^3$ is determined by a pair of distinct points on the ideal boundary. This information can be encoded by an indefinite binary quadratic form

$$Q(x, y) = (\beta x - \alpha y)(\delta x - \gamma y),\ [\alpha : \beta], [\gamma : \delta] \in P^1(\mathbb{R}) \text{ or } P^1(\mathbb{C}),$$

whose zero set, $Z(Q)$, consists of the two points at the boundary.

In $\mathcal{H}^3$, geodesic planes are determined by their boundary, a copy of $P^1(\mathbb{R})$ in $P^1(\mathbb{C})$. This boundary can be described by the zero set $Z(H)$ of an indefinite binary Hermitian form

$$H(z, w) = (\bar{z}\ \bar{w}) \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix},\ A, C \in \mathbb{R},\ B \in \mathbb{C},\ \Delta := \det(H) = AC - |B|^2 < 0,$$

$$Z(H) = \{[z : w] : H(z, w) = 0\} = \begin{cases} \{[z : 1] : |z + B/A|^2 = -\Delta/A^2\} & A \neq 0, \\ \{[1 : w] : |w + \overline{B}/C|^2 = -\Delta/C^2\} & C \neq 0, \\ \{[z : w] : z\overline{B}\bar{w} + \bar{z}Bw = 0\} & A = C = 0. \end{cases}$$

Once again we have equivariance, $g^{-1}Z(H) = Z(H^g)$, where the left action on the zero set is the Möbius action and the right action on the form is change of variable.

1.4 The geodesic flow in $\mathcal{H}^2$ and $\mathcal{H}^3$

The unit tangent bundle of $\mathcal{H}^2$, $T^1(\mathcal{H}^2) \subseteq \mathbb{C} \times \mathbb{C}$ can be identified with $PSL_2(\mathbb{R})$, explicitly by the simply transitive action

$$g \cdot (z, v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, v) = \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2} \right) = \left(g \cdot z, \frac{dg}{dz} \cdot v \right),$$

after choosing a basepoint for the action ((i, i) for instance). The geodesic flow on $T^1(\mathcal{H}^2)$ is

$$\Phi_t(z, v) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

unit-speed flow for time t starting at z in the direction v. We can forget the tangent information and consider geodesic trajectories

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} SO_2(\mathbb{R})$$

in $\mathcal{H}^2 \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$.

Similarly, in $\mathcal{H}^3$ the oriented orthonormal frame bundle can be identified with $PSL_2(\mathbb{C})$ via the derivative action (here $\eta \in \mathbb{C} + \mathbb{R}j$ is a tangent vector)

$$g \cdot (\zeta, \eta) = \big(g(\zeta), (a - g(\zeta)c)\eta(c\zeta + d)^{-1} \big) = (g(\zeta), (\zeta c + d)^{-1}\eta(c\zeta + d)^{-1}),$$

say with base point $(j, \{1, i, j\})$, the frame flow (parallel transport of a framed point $(p, \{e_1, e_2, e_3\})$ along the geodesic determined by (p, e_3)) can be written in a simliar manner

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

and we can forget the framing, looking only at points of $\mathcal{H}^3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} SU_2(\mathbb{C}).$$

Chapter 2

Badly approximable systems of linear forms and the Dani correspondence

2.1 Introduction

In this chapter, we first give a short discussion of the space of unimodular lattices in $\mathbb{R}^n$, $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$, a basic compactness criterion for subsets of lattices (Mahler's compactness criterion), and then translate between the notions of badly approximable systems of linear forms and bounded trajectories in the space of lattices (the Dani correspondence).

For more on the space of unimodular lattices, lattices in Lie groups, and arithmetic groups, see [Mor15], [Bor69], [Rag72], [PR94], [Sie89], [Ebe96].

2.2 The space of unimodular lattices and Mahler's compactness criterion

A (full) lattice $\Lambda \subseteq \mathbb{R}^n$ is a discrete additive subgroup isomorphic to $\mathbb{Z}^n$. A lattice Λ is unimodular if it has covolume one (e.g. the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ has volume one). By choosing a $\mathbb{Z}$-basis for the lattice, up to change of basis, the space of unimodular lattices is the coset space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$. In other words, we specify a unimodular lattice in $\mathbb{R}^n$ by the rows of a matrix in $SL_n(\mathbb{R})$. This space is non-compact but has finite volume (equal to $\zeta(2) \cdot \ldots \cdot \zeta(n)$ when suitably normalized [Sie89]) induced by Haar measure on $SL_n(\mathbb{R})$.

In the sequel, we will make use of the following criterion for determining whether or not a subset of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ is bounded (has compact closure).

Theorem 2.2.1 (Mahler's compactness criterion, [Mah46] Theorem 2)**.** *A subset $\Omega \subseteq SL_n(\mathbb{R})$ is precompact modulo $SL_n(\mathbb{Z})$ if and only if the corresponding lattices (given by the $\mathbb{Z}$-span of the rows* *of $\omega \in \Omega$) contain no arbitrarily short vectors. In other words*

$$\inf\{\|(x_1, \ldots, x_n)\omega\|_\infty : \omega \in \Omega, (x_1, \ldots, x_n) \in \mathbb{Z}^n \setminus \{0\}\} > 0,$$

(where $\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}$).

2.3 Compact quotients from anisotropic forms

Let $F(x) = F(x_1, \ldots, x_n)$ be an anisotropic integral n-ary form, i.e. F is a homogeneous polynomial in n variables such that

$$F(\mathbb{Z}^n) \subseteq \mathbb{Z}^n,\ F(x) \neq 0 \text{ if } \ 0 \neq x \in \mathbb{Z}^n,$$

with real and integral automorphism groups

$$Aut(F, \mathbb{R}) = \{g \in SL_n(\mathbb{R}) : F(xg) = F(x)\},\ Aut(F, \mathbb{Z}) = SL_n(\mathbb{Z}) \cap Aut(F, \mathbb{R}).$$

Lemma 2.3.1. *If F is an anisotropic integral n-ary form, then*

$$Aut(F, \mathbb{Z})\backslash Aut(F, \mathbb{R}) \subseteq SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$$

is compact.

Proof. By Mahler's criterion, we need only show that

$$\inf\{\|xg\| : g \in Aut(F, \mathbb{R}), 0 \neq x \in \mathbb{Z}^n\} > 0,$$

since $Aut(F, \mathbb{R})$ is close in $SL_n(\mathbb{R})$. Because F is anisotropic and takes on integer values, we have

$$1 \leq |F(x)| = |F(xg)|,\ 0 \neq x \in \mathbb{Z}^n,\ g \in SL_n(\mathbb{Z}).$$

Because $F(x)$ is continuous, $\|xg\|$ must be bounded away from zero and thereore $Aut(F, \mathbb{R})$ is compact modulo $SL_n(\mathbb{Z})$. $\square$

By restricting scalars, the proof above applies in other situations of interest, cf. 5.6.

2.4 Dani correspondence

Let

$$L = (L_1, \ldots, L_k) = (l_{ij})_{ij}$$

be a $(n-k) \times k$ matrix of real numbers whose columns represent the k linear forms $L_j(x) = \sum_{i=1}^{n-k} l_{ij} x_i$. We say that the system of linear forms is *badly approximable* if there exists $C > 0$ such that

$$\|x\|_\infty^{1/k} \cdot \|xL - p\|_\infty \geq C$$

for $p \in \mathbb{Z}^k$ and $0 \neq x \in \mathbb{Z}^{n-k}$.

Theorem 2.4.1 ([Dan85] Theorem 2.20)**.** *The system of linear forms L is badly approximable if and only if the trajectory*

$$\begin{pmatrix} I & L \\ 0 & I \end{pmatrix} D_t \in SL_n(\mathbb{R})$$

is bounded modulo $SL_n(\mathbb{Z})$, where $D_t = diag(e^{-t}, \ldots, e^{-t}, e^{\lambda t}, \ldots, e^{\lambda t})$ and $\lambda = \frac{n-k}{k}$.

We will be interested in the case $k = 1$, $n = 2$, but with respect to other $\mathbb{R}$-algebras and discrete subrings, namely the spaces of (free, rank two, "unimodular") $\mathcal{O}_F$-modules and associated locally symmetric spaces

$$SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes \mathbb{R}),\ SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes \mathbb{R})/K,\ K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s,$$

where F is a number field with r real and s conjugate pairs of complex embeddings, and $\mathcal{O}_F$ is the ring of integers of F.

For example, the single linear form in one variable ξx ($\xi \in \mathbb{R}$ fixed) being badly approximable is the worst approximation behavior that a real number can have with respect to the rationals: there exists $C > 0$ such that

$$|\xi - p/q| \geq C/q^2 \text{ for all } p/q \in \mathbb{Q}.$$

Under the correspondence above, this is equivalent to boundedness of the trajectory

$$SL_2(\mathbb{Z})\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}SO_2(\mathbb{R})$$

on the modular surface. In other words, consider the geodesic ray $\{\xi + e^{-2t}i : t \geq 0\} \subseteq \mathcal{H}^2$ modulo the action of $SL_2(\mathbb{Z})$.

In this setting, we will say that a vector $z = (z_\sigma)_\sigma \in \mathbb{R}^r \times \mathbb{C}^s$ (indexed by inequivalent $\sigma : F \to \mathbb{C}$) is *badly approximable* if there is a constant $C > 0$ such that

$$\|q\|_\infty \|qz - p\|_\infty \geq C \text{ for all } p \in \mathcal{O}_F, 0 \neq q \in \mathcal{O}_F,$$

where we identify $a \in F$ with the vector $(\sigma(a))_\sigma \in \mathbb{R}^r \times \mathbb{C}^s$ (i.e. via the homomorphism $F \to \mathbb{R}^r \times \mathbb{C}^s$ induced by the σ). Restricting scalars and applying Mahler's criterion gives the following.

Theorem 2.4.2 ([EGL16] Theorem 2.2)**.** *A subset $\Omega \subseteq SL_2(F \otimes \mathbb{R})$ is bounded modulo $SL_2(\mathcal{O}_F)$ if and only if the $\mathcal{O}_F$-modules $\omega \in \Omega$ (generated by the rows of ω) contain no arbitrarily short vectors, i.e.*

$$\inf\{\|(x_1, x_2)\omega\|_\infty : (0, 0) \neq (x_1, x_2) \in (\mathcal{O}_F)^2\} > 0.$$

Applying the above version of Mahler's criterion gives the following version of the Dani correspondence.

Theorem 2.4.3 ([EGL16] Proposition 3.1)**.** *A vector $z = (z_\sigma)_\sigma \in F \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$ is badly approximable if and only if the trajectory*

$$\omega_t(z) = \left\{ SL_2(\mathcal{O}_F)\left(\begin{pmatrix} 1 & z_\sigma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}\right)_\sigma \right\}$$

is bounded in $SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes \mathbb{R})$.

Our applications of the above will be discussed in chapter 5. For completeness (and flavor), we provide proofs of the previous two results from [EGL16] when F is imaginary quadratic.

Theorem 2.4.4 (Mahler's compactness criterion)**.** *Let* $\Omega \subseteq SL_2(\mathbb{C})$. *The set* $SL_2(\mathcal{O}) \cdot \Omega \subseteq SL_2(\mathcal{O})\backslash SL_2(\mathbb{C})$ *is precompact if and only if*

$$\inf\{\|Xg\|_2 : g \in \Omega,\ X = (x_1, x_2) \in \mathcal{O}^2 \setminus \{(0,0)\}\} > 0,$$

where $\|X\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$.

Proof of Mahler's criterion. Choose an integral basis for $\mathcal{O}$, say 1 and $\omega = \frac{D_F + \sqrt{D_F}}{2}$ for concreteness, where D_F is the field disciminant,

$$D_F = \begin{cases} -d, & d \equiv 3 \bmod 4 \\ -4d, & d \equiv 1, 2 \bmod 4 \end{cases}.$$

We have a homomorphism

$$\phi : \mathbb{C} \to M_2(\mathbb{R}),\ z \mapsto \begin{pmatrix} r & s \\ s\frac{D_F(1-D_F)}{4} & r + sD_F \end{pmatrix},\ z = r + s\omega,$$

taking a complex number z to the matrix of multiplication by z in the our integral basis. This extends to a homomorphism

$$\Phi : SL_2(\mathbb{C}) \to SL_4(\mathbb{R}),\ \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \mapsto \begin{pmatrix} \phi(z_1) & \phi(z_2) \\ \phi(w_1) & \phi(w_2) \end{pmatrix},$$

with $\Phi(SL_2(\mathbb{C})) \cap SL_4(\mathbb{Z}) = \Phi(SL_2(\mathcal{O}))$. Hence we obtain a closed embedding

$$\widetilde{\Phi} : SL_2(\mathcal{O})\backslash SL_2(\mathbb{C}) \to SL_4(\mathbb{Z})\backslash SL_4(\mathbb{R}).$$

One can easily verify that the bijection

$$\Psi : \mathbb{C}^2 \to \mathbb{R}^4,\ (a + b\omega, c + d\omega) \mapsto (a, b, c, d)$$

is $SL_2(\mathbb{C})$-equivariant, i.e.

$$\Psi((a + b\omega, c + d\omega)g) = (a, b, c, d)\Phi(g),\ g \in SL_2(\mathbb{C}),$$

and that the norms $\|\Psi(\cdot)\|_2$, $\|\cdot\|_2$ are equivalent on $\mathbb{C}^2$:

$$R_+\|\Psi(\cdot)\|_2 \leq \|\cdot\|_2 \leq R_-\|\Psi(\cdot)\|_2,$$

where

$$R_{\pm} = \left(\frac{2}{1 + |\omega|^2 \pm |1 + \omega^2|} \right)^{1/2}$$

are the radii of the inscribed and circumscribed circles of the ellipse $|a + b\omega|^2 = 1$. Applying the standard version of Mahler's criterion (Theorem 2.2.1 above) to $\Phi(\Omega)$ gives the result. □

Theorem 2.4.5 (Dani correspondence)**.** *For $z \in \mathbb{C}$, define*

$$\Omega_z = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathbb{C}).$$

The trajectories

$$SL_2(\mathcal{O}) \cdot \Omega_z \subseteq SL_2(\mathcal{O}) \backslash SL_2(\mathbb{C}),$$
$$\omega_z := SL_2(\mathcal{O}) \cdot \Omega_z \cdot SU_2(\mathbb{C}) \subseteq SL_2(\mathcal{O}) \backslash \mathcal{H}^3,$$

are precompact if and only if z is badly approximable.

Proof of Dani correspondence. For the proof, note that

$$\mathcal{O}^2 \cdot \Omega_z = \{(e^{-t}q, e^t(p + qz)) : t \geq 0,\ (q,p) \in \mathcal{O}^2\}.$$

Suppose z is badly approximable with $|q(qz + p)| \geq C'$ for all $p/q \in F$. If there exists $t \geq 0$ and $p/q \in F$ with $\|(e^{-t}q, e^t(qz + p))\|_2 < \sqrt{C'}$, then taking the product of the coordinates gives

$$|e^{-t}qe^t(qz + p)| = |q(qz + p)| < C',$$

a contradiction. Hence $\inf\{\|Xg\|_2 : g \in \Omega_z,\ X \in \mathcal{O}^2 \setminus \{(0,0)\}\} \geq \sqrt{C'}$ and $SL_2(\mathcal{O}) \cdot \Omega_z$ is precompact by Mahler's criterion.

If z is not badly approximable, then for every $n > 0$ there exists $p_n/q_n \in F$ such that $|q_n(q_n z + p_n)| < 1/n^2$. If t_n is such that $e^{-t_n}|q_n| = 1/n$, then $|e^{t_n}(q_n z + p_n)| = n|q_n(q_n z + p_n)| < 1/n$ and $\|(e^{-t_n}q_n, e^{t_n}(q_n z + p_n))\|_2 \leq \sqrt{2}/n$. Therefore $SL_2(\mathcal{O}) \cdot \Omega_z$ is not contained in any compact set by Mahler's criterion. The result for ω_z follows from the invariance $\|Xk\|_2 = \|X\|_2$ for $X \in \mathbb{C}^2$, $k \in SU_2(\mathbb{C})$, and the fact that the projection $SL_2(\mathcal{O}) \backslash SL_2(\mathbb{C}) \to SL_2(\mathcal{O}) \backslash \mathcal{H}^3$ is proper and closed. □

Chapter 3

Continued fractions in $\mathbb{R}$ and $\mathbb{C}$

3.1 Introduction

In this chapter we first give an overview of simple continued fractions on the real line including the relation to the geodesic flow on the modular surface (perhaps first exploited by E. Artin, [Art24]). There are innumerable books on continued fractions, including: [EW11], [Sch80], [Khi97], [Bil65], [Per54], and [Hen06]. Secondly, we give an overview of nearest integer continued fractions over the Euclidean imaginary quadratic fields, $\mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$, which go back to A. Hurwitz [Hur87]. A few of the results are new, or at least proofs could not be found in the literature. Finally, we end with an quick summary of A. L. Schmidt's continued fractions over $\mathbb{Q}(\sqrt{-1})$ (cf. [Sch75a], [Sch82], [Nak88a], [Nak90]). These continued fractions are based on partitions of the plane associated to circle packings, which we reimagine and generalize in chapter 4.

3.2 Simple continued fractions

The integers are a Euclidean domain, with respect to the usual absolute value for instance. For a pair of integers $a, b \neq 0$ we have

$$a = ba_0 + r_0,\ 0 \leq r_0 < |b|$$

$$b = r_0 a_1 + r_1,\ 0 \leq r_1 < r_0$$

$$r_0 = r_1 a_2 + r_2,\ 0 \leq r_2 < r_1$$

$$\ldots,$$

or written in matrices

$$\begin{aligned}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b \\ r_0 \end{pmatrix}\\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r_0 \\ r_1 \end{pmatrix}\\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\\ &\ldots,\end{aligned}$$

which if $(a, b) = 1$ gives

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

when the algorithm stops. Thinking of this as an algorithm on rationals (dividing by $b, r_0, r_1, \ldots$ in the first array or acting by fractional linear transformations in the second) we obtain an expression for a/b as a continued fraction

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots \frac{1}{a_n}}}} =: [a_0; a_1, \ldots, a_n].$$

Extending this to irrational numbers $\xi = \lfloor \xi \rfloor + \{\xi\} = a_0 + \xi_0$ gives a dynamical system

$$T : [0, 1) \to [0, 1),\ \xi \mapsto \{1/\xi\}$$

and infinite sequences

$$\xi_n = T^n \xi_0,\ a_{n+1} = \left\lfloor \frac{1}{T^n \xi_0} \right\rfloor,\ \begin{pmatrix} p_n & q_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

One can verify the following properties by induction or from the definitions:

$$q_n/q_{n-1} > 1\ (n \geq 2),\ q_n \geq 2^{\frac{n-1}{2}}\ (n \geq 0),\ q_n \xi - p_n = (-1)^n \xi_0 \cdot \ldots \cdot \xi_n = \frac{(-1)^n \xi_n}{q_{n-1}\xi_n + q_n},$$

$$\frac{p_n}{q_n} = a_0 - \sum_{k=1}^{n} \frac{(-1)^k}{q_k q_{k-1}}, \ \xi = \frac{p_n + p_{n-1}\xi_n}{q_n + q_{n-1}\xi_n}, \ \frac{1}{q_{n+2}} \leq |q_n \xi - p_n| \leq \frac{1}{q_{n+1}}.$$

Hence for irrational ξ we have convergence

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} =: [a_0; a_1, a_2, \ldots].$$

The branches of T^{-1} are all surjective and we have bijections

$$\mathbb{R} \setminus \mathbb{Q} \cong \mathbb{Z} \times \mathbb{N}^{\mathbb{N}}, \ \mathbb{Q} = \{[a_0; a_1, \ldots, a_n] : n \geq 0, \ a_n \neq 1 \text{ if } n \geq 1\}.$$

T is the left shift on these sequences, $T([0; a_1, a_2, \ldots]) = [0; a_2, a_3, \ldots]$.

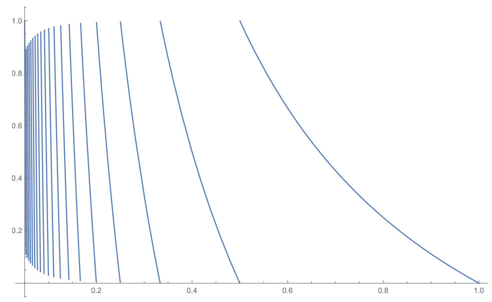

Figure 3.1: $T\xi = \{\frac{1}{\xi}\}$

3.2.1 Invertible extension and invariant measure

The map T is not invertible, but we can construct an invertible extension on $\mathcal{G} = (-\infty, -1) \times (0, 1)$

$$\widetilde{T} : \mathcal{G} \to \mathcal{G}, \ \widetilde{T}(\eta, \xi) = (1/\eta - \lfloor 1/\xi \rfloor, 1/\xi - \lfloor 1/\xi \rfloor),$$

defined piecewise by Möbius transformations acting diagonally

$$\widetilde{T}(\eta, \xi) = (g \cdot \eta, g \cdot \xi), \ \widetilde{T}^{-1}(\eta, \xi) = (h \cdot \eta, h \cdot \xi),$$

$$g = \begin{pmatrix} -\lfloor 1/\xi \rfloor & 1 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 0 & 1 \\ 1 & \lfloor -\eta \rfloor \end{pmatrix}.$$

We will think of $\mathcal{G} = (-\infty, -1) \times (0, 1)$ as a space of geodesics in hyperbolic 2-space, $\mathcal{H}^2$, and the action of $\widetilde{T}$ takes this space to itself piecewise by isometries $\text{Isom}(\mathcal{H}^2) \cong PGL_2(\mathbb{R})$, where

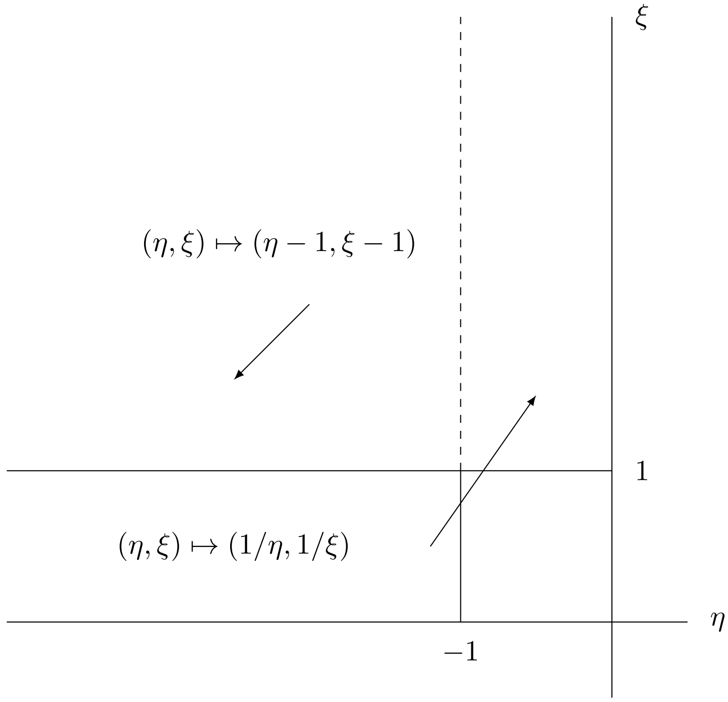

Figure 3.2: Invertible extension of the (slow) Gauss map on $(-\infty, 0) \times (1, \infty) \cup (-\infty, -1) \times (0, 1)$.

$PGL_2(\mathbb{R})$ acts on the upper half-plane by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \text{ or } \frac{a\bar{z}+b}{c\bar{z}+d}$$

according as the determinant $ad - bc$ is positive or negative.

Let $\mathcal{H}^2$ have coordinates (x, y) with area $\frac{dxdy}{y^2}$. The top of the geodesic (η, ξ) (semi-circle from η to ξ with center on the real line $x = 0$) has coordinates $\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$, so that hyperbolic area becomes $d\tilde{\mu}(\eta, \xi) := \frac{d\xi d\eta}{(\xi-\eta)^2}$ in these coordinates. This gives an isometry invariant measure on our space of geodesics, and since $\widetilde{T}$ is a bijection defined piecewise by isometries, $\tilde{\mu}$ is $\widetilde{T}$-invariant. Pushing forward to the second coordinate gives a T-invariant measure on $(0, 1)$

$$d\mu(\xi) = d\xi \int_{-\infty}^{-1} \frac{d\eta}{(\xi - \eta)^2} = \frac{d\xi}{1 + \xi}$$

which we will normalize to a probability measure by dividing by $\log 2$. This measure was known to Gauss, although more than 100 years passed before details and answers to some of Gauss' questions were given by Kuzmin [Kuz32]; see [Bre91] for some history.

3.2.2 Direct proof of ergodicity

(Following [Bil65].) For fixed $a_1, \ldots, a_n \in \mathbb{N}$, we have the cylinder set

$$\Delta_n = \{\psi(t) = \frac{p_n + p_{n-1}t}{q_n + q_{n-1}t} : 0 \leq t < 1\}$$

which is the half-open interval between p_n/q_n and $\frac{p_n+p_{n-1}}{q_n+q_{n-1}}$ (oriented depending on the parity of n). These cylinder sets generate the Borel σ-algebra. If λ is Lebesgue measure, then we have (bar indicating conditional probability)

$$\lambda(T^{-n}[s, t) \mid \Delta_n) = \frac{\psi(t) - \psi(s)}{\psi(1) - \psi(0)} = (t - s)\frac{q_n(q_n + q_{n-1})}{(q_n + q_{n-1}s)(q_n + q_{n-1}t)} = (t - s)C,$$

where $1/2 \leq C \leq 2$. Hence there exists (a different) $C > 0$ such that

$$\frac{1}{C}\mu(A) \leq \mu(T^{-n}A \mid \Delta_n) \leq C\mu(A)$$

for measurable A.

Considering T-invariant sets of positive measure, we have

$$\begin{aligned} T^{-1}A = A &\Rightarrow \frac{1}{C}\mu(A) \le \mu(A \mid \Delta_n) \\ \mu(A) > 0 &\Rightarrow \frac{1}{C}\mu(\Delta_n) \le \mu(\Delta_n \mid A) \\ &\Rightarrow \frac{1}{C}\mu(B) \le \mu(B \mid A) \text{ for any measurable} B \\ A^c = B &\Rightarrow \mu(A^c) = 0,\ \mu(A) = 1. \end{aligned}$$

Hence μ is ergodic.

3.2.3 Consequences of ergodicity

We can apply the following ergodic theorem to various functions to obtain almost everywhere statistics for continued fractions. References for any ergodic theory used here include [Wal82], [Bil65], [EW11].

Theorem 3.2.1. *Supose (X, T, μ) is a measure preserving system and $f \in L^1(\mu)$, then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{n-1} f \circ T^n = f^*$$

exists almost everywhere and

$$\int_X f d\mu = \int_X f^* d\mu$$

(i.e. f^* is the conditional expectation of f with respect to the σ-algebra of T-invariant sets). In particular, if the system is ergodic then f^* is constant almost everywhere,

$$f^* = \int_X f d\mu.$$

For instance:

- If f is the indicator of the interval $(\frac{1}{k+1}, \frac{1}{k})$, we get

$$\mathbb{P}(a_n = k) = \lim_{N\to\infty} \frac{1}{N} |\{i : a_i = k,\ 1 \le i \le N\}| = \frac{1}{\log 2} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{d\xi}{1+\xi} = \frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right),$$

k	1	2	3	4	5	6	7
$\mathbb{P}(a_n = k)$	41.58%	16.99%	9.31%	5.89%	4.06%	2.97%	2.27%

- Taking f to be $\sum_k \log k \cdot \mathbf{1}_{(\frac{1}{k+1},\frac{1}{k})}$, we get (almost everywhere)

$$\lim_{N\to\infty} \left(\prod_{n=1}^{N} a_n\right)^{1/N} = \exp\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log a_n\right) = \exp\left(\frac{1}{\log 2}\sum_k \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{\log k}{1+\xi} d\xi\right)$$
$$= \prod_k \left(\frac{(k+1)^2}{k(k+2)}\right)^{\frac{\log k}{\log 2}} = 2.6854520010\dots$$

- With $f_M = \sum_{k\le M} k \cdot \mathbf{1}_{(\frac{1}{k+1},\frac{1}{k})}$ and taking $M \to \infty$ we get

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{\infty} a_n = \infty.$$

With a little more work we can show

$$\lim_{n\to\infty} \frac{1}{n}\log q_n = \frac{1}{\log 2}\cdot\frac{\pi^2}{12} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\log|\xi - p_n/q_n| = -\frac{1}{\log 2}\cdot\frac{\pi^2}{6}.$$

We have

$$\begin{pmatrix} \xi \\ 1 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} 1 \\ T^n\xi \end{pmatrix}, \; T^n\xi = (-1)^{n-1}\frac{q_n\xi - p_n}{q_{n-1}\xi - p_{n-1}},$$

so that

$$\prod_{k=0}^{n-1} T^k\xi = (-1)^n(q_{n-1}\xi - p_{n-1}) = |q_{n-1}\xi - p_{n-1}|.$$

Hence, from the list of properties in the first section, we have

$$\frac{1}{q_{n+1}} \le \prod_{k=0}^{n-1} T^k\xi \le \frac{1}{q_n}.$$

Taking logarithms and letting $n \to \infty$ we get

$$-\lim_{n\to\infty} \frac{1}{n}\log q_{n+1} \le \lim_{n\to\infty} \sum_{k=0}^{n-1} \log T^k\xi \le -\lim_{n\to\infty} \frac{1}{q_n}\log q_n,$$

so that

$$\lim_{n\to\infty} \frac{1}{n}\log q_n = -\frac{1}{\log 2}\int_0^1 \frac{\log\xi}{1+\xi}d\xi = \frac{1}{\log 2}\sum_{k=0}^{\infty}(-1)^{k+1}\int_0^1 \xi^k \log\xi \; d\xi$$
$$= \frac{1}{\log 2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{1}{\log 2}\cdot\frac{\pi^2}{12}.$$

Since $\frac{1}{q_nq_{n+2}} \leq |\xi - \frac{p_n}{q_n}| \leq \frac{1}{q_nq_{n+1}}$, the above gives

$$\lim_{n\to\infty} \frac{1}{n} \log|\xi - p_n/q_n| = -\frac{1}{\log 2} \cdot \frac{\pi^2}{6}.$$

Finally, a result on the normalized error $\theta_n(\xi) = q_n|q_n\xi - p_n|$ (assuming $\widetilde{T}$ is ergodic).

Proposition 3.2.2 ([Hen06] Theorem 4.1)**.** *For μ almost every ξ, we have*

$$\lim_{N\to\infty} \frac{1}{N}|\{1 \leq n \leq N : \theta_n(\xi) \leq t\}| = \begin{cases} \frac{t}{\log 2} & 0 \leq t \leq 1/2 \\ 1 + \frac{1-t+\log t}{\log 2} & 1/2 \leq t \leq 1 \\ 1 & t \geq 1 \end{cases}.$$

Proof. Note that

$$\widetilde{T}^n(-\infty, \xi) = (-q_n/q_{n-1}, T^n\xi) = (-[a_n; a_{n-1}, \ldots, a_1], [0; a_{n+1}, a_{n+2}, \ldots])$$
$$\theta_n(\xi) = \frac{1}{1/T^n\xi + q_{n-1}/q_n} = \frac{1}{[a_{n+1}; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1]},$$

so that $\theta_n(\xi) \leq t$ iff $\frac{1}{1/\xi' - 1/\eta'} \leq t$ where $(\eta', \xi') = \widetilde{T}^n(-\infty, \xi)$. Let $\mathcal{G}(c) = \{(\eta, \xi) \in \mathcal{G} : 1/\xi - 1/\eta \geq c\}$ Then for $\epsilon > 0$ and n large, we have

$$\widetilde{T}^n(\eta, \xi) \in \mathcal{G}(1/t + \epsilon) \Rightarrow \widetilde{T}^n(-\infty, \xi) \in \mathcal{G}(1/t) \Rightarrow \widetilde{T}^n(\eta, \xi) \in \mathcal{G}(1/t - \epsilon).$$

The measure of $\mathcal{G}(c)$ with respect to $\frac{d\xi d\eta}{(\xi-\eta)^2 \log 2}$ for $c \geq 1$ is

$$\begin{cases} \frac{1}{\log 2}\left(1 - \frac{1}{c} + \log 2 - \log c\right) & 1 \leq c \leq 2 \\ \frac{1}{c \log 2} & c \geq 2 \end{cases},$$

which gives the result when $t = 1/c$. See Figure 3.3 for a graph of the distribution. □

3.2.4 Geodesic flow on the modular surface and continued fractions

(Following chapter 9 of [EW11] in spirit if not detail.) The group $SL_2(\mathbb{R})$ acts transitively on the upper half-plane by fractional linear transformations, and the stabilizer of $z = i$ is $SO_2(\mathbb{R})$. Hence, as a homogeneous space, $\mathcal{H}^2 \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Moreover $SL_2(\mathbb{R})$ acts as orientation perserving isometries and in fact $PSL_2(\mathbb{R}) \cong \mathrm{Isom}^+(\mathcal{H}^2)$.

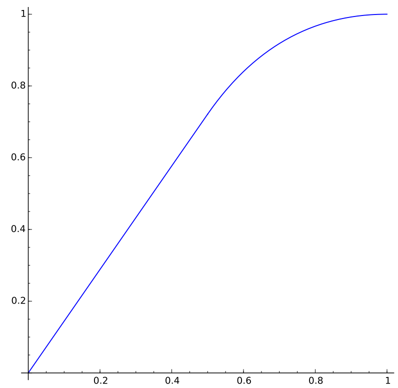

Figure 3.3: Distribution of the normalized error for almost every ξ.

We can identify $PSL_2(\mathbb{R})$ with $T^1(\mathcal{H}^2)$, the unit tangent bundle, as follows. Let $(z, v) \in T^1(\mathcal{H}^2) \subseteq \mathcal{H}^2 \times \mathcal{C}$, where $z = x + yi$ is in the upper half-plane and v has norm 1 in the hyperbolic metric, i.e. if $v = v_1 + v_2 i$, then $\langle v, v\rangle_z := v_1 v_2 / y^2 = 1$. Define the derivative action of $SL_2(\mathbb{R})$ on $T^1(\mathcal{H}^2)$ by

$$g \cdot (z, v) = (g(z), g'(z)v) = \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right).$$

One can verify that this action is isometric and transitive with kernel ± 1, so that we get the identification $PSL_2(\mathbb{R}) \cong T^1(\mathcal{H}^2)$.

Under the NAK decomposition, with

$$p = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \in NA, \;\; k = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in K,$$

we have $pk(i, i) = (x + yi, yie^{i\theta})$.

Associating the identity with the point $(z, v) = (i, i) \in T^1(\mathcal{H}^2)$, the *geodesic flow* $\Phi_t : T^1(\mathcal{H}^2) \to T^1(\mathcal{H}^2)$ is right multiplication by $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. It takes a point (z, v) along a unit speed geodesic in the direction v for time t, as can be checked along the imaginary axis and extended via the isometric action.

The group $\Gamma = SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$ and the quotient $\Gamma\backslash SL_2(\mathbb{R})$ has finite volume, equivalently, the quotient $\Gamma\backslash\mathcal{H}^2$ has finite hyperbolic area (namely $\pi/3$). We identify

$\Gamma\backslash SL_2(\mathbb{R})$ with the unit tangent bundle of the modular surface.

We now want to relate continued fractions (or its invertible extension) with the geodesic flow on the modular surface. There is a slight complication because continued fractions are defined over $GL_2(\mathbb{Z})$. Consider the following subsets of $T^1(\mathcal{H}^2)$

$$\begin{aligned} \mathcal{C}^+ &= \{(z,v) \in \overrightarrow{\eta\xi} : (\eta,\xi) \in \mathcal{G},\ z \in i\mathbb{R}\}, \\ \mathcal{C}^- &= \{(z,v) \in \overrightarrow{(-\eta)(-\xi)} : (\eta,\xi) \in \mathcal{G},\ z \in i\mathbb{R}\}, \\ \mathcal{C} &= \mathcal{C}^+ \cup \mathcal{C}^-, \end{aligned}$$

i.e. those points and directions of the intersection of elements of $\mathcal{G}$ with the imaginary axis (and their reflected images). Let $\pi : T^1(\mathcal{H}^2) \to T^1(\Gamma\backslash\mathcal{H}^2)$ be the natural projection.

Proposition 3.2.3. *If $(z,v) = (\eta,\xi) \in \pi(\mathcal{C}^+)$, the next return of the geodesic flow to $\pi(\mathcal{C})$ is in $\pi(\mathcal{C}^-)$ with coordinates $-\widetilde{T}(\eta,\xi)$, and similarly for $(z,v) = (-\eta,-\xi) \in \pi(\mathcal{C}^-)$, the next return of the geodesic flow to $\pi(\mathcal{C})$ is in $\pi(\mathcal{C}^+)$ with coordinates $\widetilde{T}(\eta,\xi)$. In other words, the map*

$$S : \mathcal{G} \cup -\mathcal{G},\ S(\eta,\xi) = -\widetilde{T}(\eta,\xi),\ S(-\eta,-\xi) = \widetilde{T}(\eta,\xi),$$

is the first return of the geodesic flow to the cross section $\pi(\mathcal{C})$.

Proof. Proof by picture. Applying $-1/z$ to $(\eta,\xi) \in (-\infty,-1) \times (0,1)$ gives $(-1/\eta,-1/\xi) \in (0,1) \times (-\infty,-1)$. Following the geodesic flow to the next intersection with $\pi(\mathcal{C})$ corresponds to translation by $\lfloor 1/\xi \rfloor$ where we end up in $-\mathcal{G}$ with coordinates $(\lfloor 1/\xi\rfloor - 1/\eta, \lfloor 1/\xi\rfloor - 1/\xi) = -\widetilde{T}(\eta,\xi)$. Similarly for the other case. $\square$

To complete the picture of how continued fractions fit into the geodesic flow, we need to consider the *return time* $r(\eta,\xi)$, $r(-\eta,-\xi)$ to the cross section. Here is a more general construction, the *special flow* under a function.

Proposition 3.2.4. *Suppose (X,T,μ) is a measure preserving system, and $f : X \to (0,\infty)$ is μ-measurable. Let $X_f = \{(x,s) : 0 \le s < f(x)\}$ and for each $t \ge 0$ let $\phi_t : X_f \to X_f$ be defined by*

$$\phi_t(x,s) = \left(T^n x, s + t - \sum_{k=0}^{n-1} f(T^k x)\right)$$

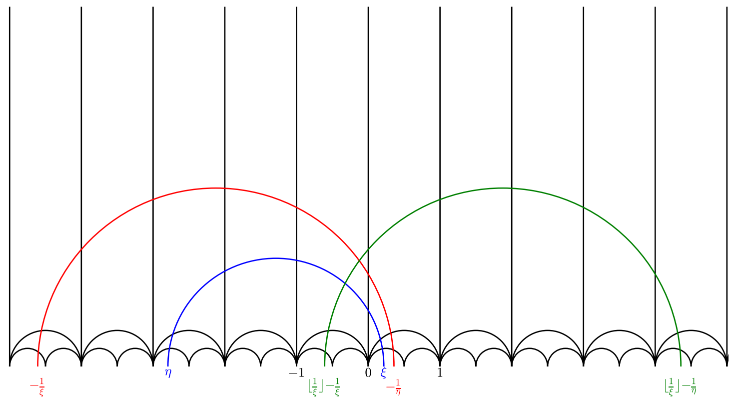

Figure 3.4: Visualizing S as first return of the geodesic flow to $\pi(\mathcal{C})$.

where n is the least non-negative integer such that $0 \leq s+t < \sum_{k=0}^{n} f(T^k x)$. Let $\mu_f = \mu \times \lambda$ restricted to X_f. Then:

- *μ_f is ϕ_t-invariant for all $t \geq 0$,*
- *$\mu_f(X_f) < \infty \Leftrightarrow f \in L^1(X, \mu)$,*
- *If $\mu_f(X_f) < \infty$, then (X, T, μ) is ergodic $\Leftrightarrow$ the flow $\{\phi_t\}_{t \geq 0}$ is ergodic.*

Let $r : \mathcal{G} \cup -\mathcal{G} \to (0, \infty)$ be the return time of the associated $(z, v) \in \pi(\mathcal{C})$ to $\pi(\mathcal{C})$, $X_r = (\mathcal{G} \cup -\mathcal{G})_r$, $S(\eta, \xi)$ as above, and ϕ the special flow associated to r. Let $\Sigma = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$ be the modular surface, $T^1(\Sigma) = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ its unit tangent bundle, and Φ the geodesic flow (right multiplication by $\mathrm{diag}(e^{t/2}, e^{-t/2})$). We have

Proposition 3.2.5. *The following diagram commutes*

$$\begin{tikzcd} X_r \arrow[r, "\phi_t"] \arrow[d] & X_r \arrow[d] \\ T^1(\Sigma) \arrow[r, "\Phi_t"] & T^1(\Sigma) \end{tikzcd}$$

where the arrows on the left and right are $((y, x), s) \mapsto \Phi_s(z, v)$. Moreover, the measure μ_r for the special flow under the return time (constructed from Gauss measure and Lebesgue measure) is the pullback (up to a multiplicative constant) of the Φ-invariant measure $\frac{d\theta dx dy}{y^2}$ on the unit tangent bundle (Haar measure on $PSL_2(\mathbb{R})$).

Proof. That the above commutes follows from our earlier discussion, so we now consider the measures involved. We have two coordinate systems on $PSL_2(\mathbb{R}) \cong T^1(\mathcal{H}^2)$, thinking of a point $z = x + yi$ and direction $v = iye^{i\theta}$ at z,

$$g(x, y, \theta) = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

discussed above, and another obtained by associating to (z, v) the geodesic (η, ξ) it determines and the distance/time one must go from the "top" of the geodesic

$$g(\eta, \xi, t) = \frac{1}{\sqrt{|\eta - \xi|}} \begin{pmatrix} \max\{\xi, \eta\} & \min\{\xi, \eta\} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\epsilon} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

with $\epsilon = 0$ if $\xi > \eta$ and $\epsilon = 1$ otherwise. One can think of this as taking the geodsic $\overrightarrow{0\infty}$ with marked point i, mapping it to the geodesic $\overrightarrow{\eta\xi}$ with marked point at the top, and flowing for the required time. We have an invariant measure on each of these

$$d\mu dt = \frac{d\eta d\xi}{(\xi - \eta)^2} dt, \ \frac{dxdy}{y^2} d\theta$$

and we would like to show that these are the same (perhaps up to a constant). Assume $\xi > \eta$ (the other case being similar). We have

$$\frac{1}{\sqrt{|\eta - \xi|}} \begin{pmatrix} \xi & \eta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot (i, i) = (x + yi, iye^{i\theta})$$

where

$$x = \frac{\xi e^t + \eta e^{-t}}{e^t + e^{-t}}, \ y = \frac{\xi - \eta}{e^t + e^{-t}}, \ \theta = \arctan\left(\frac{1}{\sinh t}\right).$$

Some computation shows

$$\frac{dxdyd\theta}{y^2} = \left(\frac{e^t + e^{-t}}{\xi - \eta}\right)^2 \begin{vmatrix} \frac{e^t}{e^t+e^{-t}} & \frac{e^{-t}}{e^t+e^{-t}} & * \\ \frac{1}{e^t+e^{-t}} & \frac{-1}{e^t+e^{-t}} & * \\ 0 & 0 & \frac{-1}{\cosh t} \end{vmatrix} = 2\frac{d\eta d\xi dt}{(\xi - \eta)^2},$$

as desired. $\square$

We can compute the return time from the above,

$$r(\pm\eta, \pm\xi) = \frac{1}{2}\log\left(\frac{1-\eta\lfloor 1/\xi\rfloor}{1-\xi\lfloor 1/\xi\rfloor}\right),$$

and its integral must be

$$\int_{\mathcal{G}\cup-\mathcal{G}} r(\eta,\xi)\frac{d\xi d\eta}{(\xi-\eta)^2} = \frac{\pi^2}{3}$$

since the total volume of $T^1(\Sigma)$ is $2\pi^2/3$. (I couldn't compute this integral, and neither could Mathematica, but it agrees numerically within the estimated error.)

3.2.5 Mixing of the geodesic flow

Mixing of the geodesic flow on the modular surface is implied by the following "decay of matrix coefficients" theorem, applied to the Hilbert space $\mathcal{H} = L_0^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}))$ (L^2 functions f with $\int f = 0$) and $g = \mathrm{diag}(e^{t/2}, e^{-t/2})$.

Theorem 3.2.6 (Howe-Moore, cf. [Mor15], §11.2, or [Zim84], Theorem 2.2.20)**.** *Suppose $\rho : SL_2(\mathbb{R}) \to \mathcal{U}(\mathcal{H})$ is a unitary representation with no non-zero fixed vectors, i.e. $\{v \in \mathcal{H} : \rho(G)v = v\} = \{0\}$. Then for all $v, w \in \mathcal{H}$, $\langle\rho(g)v, w\rangle \to 0$ as $g \to \infty$ (i.e. for any $\epsilon > 0$, there exists $K \subseteq G$ compact such that for $g \notin K$, $\langle\rho(g)v, w\rangle < \epsilon$).*

Proof. We first note that any $g \in SL_2(\mathbb{R})$ can be written uniquely as KA^+K (Cartain decomposition) where K is $SO_2(\mathbb{R})$ and A^+ is the collection of $\mathrm{diag}(a, a^{-1})$, $a > 0$. (Proof: diagonalize the quadratic form $\|gx\|^2$.) Let $g_n \to \infty$, $g_n = k_n a_n l_n$, and for $v, w \in \mathcal{H}$, let $\tilde{v}$, $\tilde{w}$ be weak limits of $\rho(l_n)v$, $\rho(k_n^{-1})w$. We have

$$\begin{aligned}&\langle\rho(k_n a_n l_n)v, w\rangle - \langle\rho(a_n)\tilde{v}, \tilde{w}\rangle \\ &= \langle\rho(a_n)(\rho(l_n)v - \tilde{v}), \rho(k_n^{-1})w\rangle + \langle\rho(a_n), \rho(k_n^{-1})w - \tilde{w}\rangle \to 0.\end{aligned}$$

Therefore we may assume $g_n = a_n = \mathrm{diag}(t_n, 1/t_n)$ with $t_n \to \infty$.

Let

$$u_s^+ = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},\ u_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Then

$$a_n^{-1}u_s^+a_n = u_{s/t_n^2}^+ \to 1,\ a_nu_s^-a_n^{-1} = u_{s/t_n^2}^- \to 1.$$

Let $v, w \in \mathcal{H}$, and let E be a weak limit of $\rho(a_n)$ (diagonal argument on an orthonormal basis). We want to show that $E = 0$ (in particular, $\langle \rho(a_n)v, w\rangle \to \langle Ev, w\rangle = 0$). We have

$$\langle \rho(u_s^+)Ev, w\rangle = \lim_n \langle \rho(u_s^+)\rho(a_n)v, w\rangle = \lim_n \langle \rho(a_n)\rho(a_n^{-1}u_s^+a_n)v, w\rangle = \langle Ev, w\rangle,$$

and similarly $\langle \rho(u_s^-)E^*v, w\rangle = \langle E^*v, w\rangle$. Also, $\langle \rho(a_n)v, \rho(a_m)v\rangle = \langle \rho(a_m^{-1})v, \rho(a_n^{-1})w\rangle$ (since A is Abelian), so that $EE^* = E^*E$, i.e. E is normal. If $E \neq 0$, then $EE^* \neq 0$ and there is some $v \in \mathcal{H}$ such that $w = E^*Ev = EE^*v \neq 0$. This w is invariant under $\{u_s^\pm\}_s$, which generates $SL_2(\mathbb{R})$, contradicting the assumption that π has no non-zero invariant vectors. Hence $E = 0$. □

3.3 Nearest integer continued fractions over the Euclidean imaginary quadratic fields

A natural generalization of continued fractions to complex numbers over appropriate discrete subrings $\mathcal{O}$ of $\mathbb{C}$, in particular over $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, was introduced by A. Hurwitz, [Hur87]. Let $K = K_d = \mathbb{Q}(\sqrt{-d})$, $d > 0$ a square-free integer, be an imaginary quadratic field and $\mathcal{O} = \mathcal{O}_d$ the ring of integers of K. For $d = 1, 2, 3, 7, 11$ the $\mathcal{O}_d$ are Euclidean with respect to the usual norm $|z|^2 = z\bar{z}$, noting that the collection of disks $\{z \in \mathbb{C} : |z - r| < 1\}_{r\in\mathcal{O}}$ cover the plane (Figure 3.5), and in fact are the only d for which $\mathcal{O}_d$ is Euclidean with respect to any function (cf. [Lem95] §4). Consider the open Voronoï cell for $\mathcal{O}_d \subseteq \mathbb{C}$, the collection of points closer to zero than to any other lattice point, along with a subset $\mathcal{E}$ of the boundary, so that we obtain a strict fundamental domain for the additive action of $\mathcal{O}$ on $\mathbb{C}$,

$$V = V_d = \{z \in \mathbb{C} : |z| < |z - r|,\ r \in \mathcal{O}\} \cup \mathcal{E},\ \mathcal{E} \subseteq \partial V.$$

For the Euclidean values of d, and only for these values, V_d is contained in the open unit disk. The regions V_d are rectangles for $d = 1, 2$ and hexagons for $d = 3, 7, 11$; see Figure 3.5. For $z \in \mathbb{C}$, we denote by $\lfloor z \rceil \in \mathcal{O}$ and $\{z\} \in V$ the nearest integer and remainder, uniquely satisfying

$$z = \lfloor z \rceil + \{z\}.$$

We now restrict ourselves to Euclidean K to describe the continued fraction algorithm and applications.

We have an almost everywhere defined map $T = T_d : V_d \to V_d$ given by $T(z) = \{1/z\}$. For $z \in \mathbb{C}$ define sequences $a_n \in \mathcal{O}$, $z_n \in V$, for $n \geq 0$:

$$a_0 = \lfloor z \rfloor, z_0 = z - a_0 = \{z\},$$
$$a_n = \left\lfloor \frac{1}{z_{n-1}} \right\rfloor, \; z_n = \left\{ \frac{1}{z_{n-1}} \right\} = \frac{1}{z_{n-1}} - a_n = T^n(z_0).$$

In this way, we obtain a continued fraction expansion for $z \in \mathbb{C}$,

$$z = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0; a_1, a_2, \dots],$$

where the expansion is finite for $z \in K$. The convergents to z will be denoted by

$$\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n],$$

where p_n, q_n are defined by

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Here are a few easily verified algebraic properties that will be used below:

$$q_n z - p_n = (-1)^n z_0 \cdot \ldots \cdot z_n, \; z = \frac{p_n + z_n p_{n-1}}{q_n + z_n q_{n-1}},$$
$$z - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2 (z_n^{-1} + q_{n-1}/q_n)}, \; \frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}}.$$

The first equality proves convergence $p_n/q_n \to z$ for irrational z and gives a rate of convergence exponential in n. A useful parameter is $\rho = \rho_d$, the radius of the smallest circle around zero containing V_d,

$$\rho_d = \frac{\sqrt{1+d}}{2}, \; d = 1, 2, \; \rho_d = \frac{1+d}{4\sqrt{d}}, \; d = 3, 7, 11.$$

We note that $|a_n| \geq 1/\rho_d$ for $n \geq 1$, which is easily verified for each d.

Taking the transpose of the matrix expression above, we have the equality

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1} + \frac{1}{\dots + \frac{1}{a_1}}} \stackrel{alg.}{=} [a_n; a_{n-1}, \dots, a_1]$$

as rational numbers (indicated by the overset “alg.”), but this does not hold at the level of continued fractions, i.e. the continued fraction expansion of q_n/q_{n-1} is not necessarily $[a_n; a_{n-1}, \ldots, a_1]$. See Figure 3.6 for the distribution of q_{n-1}/q_n, for 5000 random numbers and $1 \leq n \leq 10$, over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. The bounds $|q_{n+2}/q_n| \geq 3/2$ are proved in [Hen06] and [Dan17] for $d = 1$ and 3 respectively.

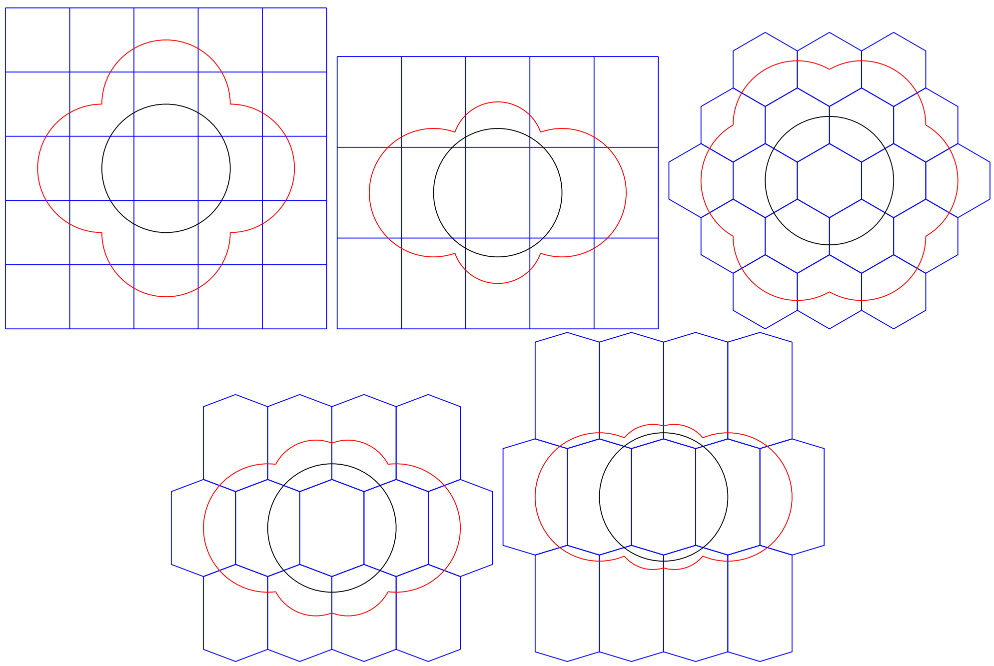

Figure 3.5: ∂V and translates (blue), $\partial(V^{-1})$ (red), and unit circle (black) for $d = 1, 2, 3, 7, 11$.

Monotonicity of the denominators q_n was shown by Hurwitz [Hur87] for $d = 1, 3$, Lunz [Lun36] for $d = 2$, and stated without proof in [Lak73] for $d = 1, 2, 3, 7, 11$. As this is a desirable property to establish, we outline the proof for the cases $d = 7, 11$ in an appendix. The proofs are unenlightening and follow the outline for the simpler cases $d = 1, 3$ in [Hur87].

Proposition 3.3.1. *For any $z \in \mathbb{C}$, the denominators of the convergents p_n/q_n are strictly increasing in absolute value, $|q_{n-1}| < |q_n|$.*

Proof. See appendix 3.3.1. ∎

For each of the Euclidean imaginary quadratic fields K there is a constant $C > 0$ such that

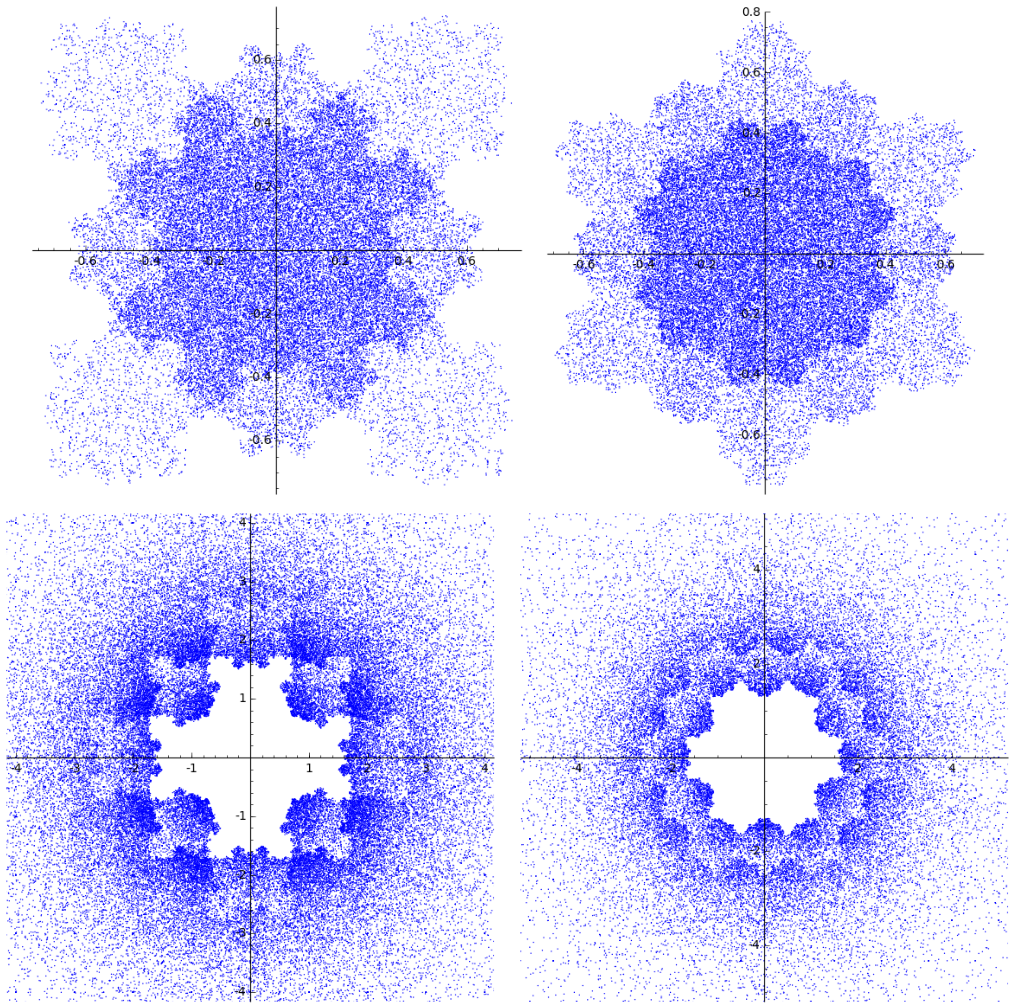

Figure 3.6: The numbers q_{n-1}/q_n and q_n/q_{n-1}, $1 \leq n \leq 10$, for 5000 randomly chosen z over $\mathbb{Q}(\sqrt{-1})$ (left) and $\mathbb{Q}(\sqrt{-3})$ (right).

for any $z \in \mathbb{C}$ there are infinitely many solutions $p/q \in K$, $(p,q) = 1$ to

$$|z - p/q| \leq C/|q|^2, \tag{$\dagger$}$$

by a pigeonhole argument for instance (cf. [EGM98] Chapter 7, Proposition 2.6). The smallest such C are $1/\sqrt{3}$, $1/\sqrt{2}$, $1/\sqrt[4]{13}$, $1/\sqrt[4]{8}$, and $2/\sqrt{5}$ for $d = 1, 2, 3, 7, 11$ respectively (for references, see the Introduction to [Vul95b], and simple geometric proofs for $d = 1, 2$ can be found in chapter 4 of this document). We can obtain rational approximations with a specific C satisfying inequality ($\dagger$) using the nearest integer algorithms described above. The best constants coming from the nearest integer convergents, $\sup_{z,n}\{|q_n|^2|z - p_n/q_n|\}$, can be found in Theorem 1 of [Lak73].

Proposition 3.3.2. *For $z \in \mathbb{C} \setminus K$, the convergents p_n/q_n satisfy*

$$|z - p_n/q_n| \leq \frac{1}{(1/\rho - 1)|q_n|^2},$$

i.e. we can take $p/q = p_n/q_n$ and $C = \frac{\rho}{1-\rho}$ in the inequality ($\dagger$).

Proof. Using simple properties of the algorithm and the bounds $1/z_n \in V^{-1}$, $|q_{n-1}/q_n| \leq 1$, we have

$$|z - p_n/q_n| = \frac{1}{|q_n|^2|z_n^{-1} + q_{n-1}/q_n|} \leq \frac{1}{|q_n|^2(1/\rho - 1)}.$$

□

We say z is *badly approximable* over K if there is a $C' > 0$ such that

$$|z - p/q| \geq C'/|q|^2$$

for all $p/q \in K$, i.e. z is badly approximable if the exponent of two on $|q|$ is the best possible in the inequality ($\dagger$). In chapter 5, we will show that the badly approximable numbers are characterized by the boundedness of the partial quotients in the nearest integer continued fraction expansion, analogous to the well-known fact for simple continued fractions over the real numbers. To this end, we need to be able to compare any rational approximation of z to those coming from the continued fraction expansion. While the convergents are not necessarily the best approximations (e.g. [Lak73]), they aren't so bad.

Lemma 3.3.3. *There are effective constants $\alpha = \alpha_d > 0$ such that for any irrational z with convergents p_n/q_n and rational p/q with $|q_{n-1}| < |q| \leq |q_n|$ we have*

$$|q_n z - p_n| \leq \alpha |qz - p|.$$

Proof. Write p/q in terms of the convergents p_n/q_n, p_{n-1}/q_{n-1} for some $s, t \in \mathcal{O}$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} p_n s + p_{n-1} t \\ q_n s + q_{n-1} t \end{pmatrix}.$$

If $s = 0$, then $p/q = p_{n-1}/q_{n-1}$, impossible by the assumption $|q_{n-1}| < |q|$. If $t = 0$, then $p/q = p_n/q_n$ and the result is clear with $\alpha = 1$. We may therefore assume $|s|, |t| \geq 1$. We have

$$\left| z - \frac{p}{q} \right| \geq \left| \left| \frac{p_n}{q_n} - \frac{p}{q} \right| - \left| z - \frac{p_n}{q_n} \right| \right| = \left| \left| \frac{t}{q q_n} \right| - \left| z - \frac{p_n}{q_n} \right| \right|,$$

noting that $t = (-1)^n (p q_n - p_n q)$ by inverting the matrix relating p, q, s, and t.

Define δ by $|t| = \delta |q_n|^2 |z - p_n/q_n|$, so that

$$\frac{1}{|\delta - |q/q_n||} |qz - p| \geq |q_n z - p_n|.$$

If $\delta > 1$, then we have our α. A lower bound for δ is

$$\delta = \frac{|t|}{|q_n|^2 |z - p_n/q_n|} \geq |t| \inf_{z,n} \{(|q_n||q_n z - p_n|)^{-1}\}.$$

The infimum above is calculated in [Lak73], Theorem 1, where it is found to be

$$\inf_{z,n} \{(|q_n||q_n z - p_n|)^{-1}\} = \begin{cases} 1 & d = 1, \\ \sqrt{\frac{486 - \sqrt{3}}{786}} = 0.78493\ldots & d = 2, \\ \sqrt{\frac{7 + \sqrt{21}}{7}} = 1.28633\ldots & d = 3, \\ \sqrt{\frac{2093 - 9\sqrt{21}}{2408}} = 0.92307\ldots & d = 7, \\ \sqrt{\frac{30 - 8\sqrt{5} - 5\sqrt{11} + 3\sqrt{55}}{50}} = 0.59627\ldots & d = 11. \end{cases}$$

The smallest integers of norm greater than one in $\mathcal{O}_d$ have absolute values of $\sqrt{2}$ (for $d = 1, 2, 7$) and $\sqrt{3}$ (for $d = 3, 11$). Multiplying these potential values of $|t|$ by the above constants gives values of δ greater than one, so that $|\delta - |q/q_n|| \geq |\delta - 1|$ is bounded away from zero. Hence we are left to explore those rationals p/q with $|t| = 1$.

For general t we have

$$\begin{aligned} qz - p &= q\frac{p_n + z_n p_{n-1}}{q_n + z_n q_{n-1}} - p = (q_n s + t q_{n-1})\frac{p_n + z_n p_{n-1}}{q_n + z_n q_{n-1}} - (p_n s + t p_{n-1}) \\ &= \frac{(-1)^n (s z_n - t)}{q_n + z_n q_{n-1}} \end{aligned}$$

and

$$q_n z - p_n = \frac{(-1)^n z_n}{q_n + z_n q_{n-1}},$$

and we want $\alpha > 0$ such that

$$|q_n z - p_n| \leq \alpha |qz - p|.$$

Substituting the above we have

$$\begin{aligned} |q_n z - p_n| \leq \alpha |qz - p| &\Longleftrightarrow \frac{|z_n|}{|q_n + z_n q_{n-1}|} \leq \alpha \frac{|z_n s - t|}{|q_n + z_n q_{n-1}|} \\ &\Longleftrightarrow \frac{1}{|s - t/z_n|} \leq \alpha. \end{aligned}$$

If $|s - t/z_n| < 1/2$ and $|t| = 1$, then $s/t = a_{n+1}$ since $s/t \in \mathcal{O}$ is the nearest integer to $1/z_n$. However (with $|q_{n-1}| < |q| \leq |q_n|$, $q = s q_n + t q_{n-1}$),

$$\left|\frac{q_{n+1}}{q_n}\right| = \left|a_{n+1} + \frac{q_{n-1}}{q_n}\right| = \left|\frac{s}{t} + \frac{q_{n-1}}{q_n}\right| = \left|s + t\frac{q_{n-1}}{q_n}\right| = \left|\frac{q}{q_n}\right| \leq 1,$$

and we obtain a contradiction if $|s - t/z_n| < 1/2$ and $|t| = 1$. Hence when $|t| = 1$ we can take $\alpha = 2$.

In summary, we can take

$$\alpha_d = \begin{cases} 2.41421\ldots & d = 1 \\ 9.08592\ldots & d = 2 \\ 2 & d = 3 \\ 3.27419\ldots & d = 7 \\ 30.51490\ldots & d = 11 \end{cases},$$

taking the maximum of 2 (covering the case $|t| = 1$) and the bound on $\frac{1}{\delta - |q/q_n|}$ for $|t| > 1$. $\square$

No attempt was made to optimize the value of α in the lemma. The above result for $d = 1, 3$ and $\alpha = 1$ is contained in Theorem 2 of [Lak73]. Another proof for $d = 1$ and $\alpha = 5$ is Theorem 5.1 of [Hen06], and a proof for $d = 3$, $\alpha = 2$ can be found in [Dan17].

3.3.1 Appendix: Monotonicity of denominators for $d = 7, 11$

The purpose of this appendix is to prove Proposition 3.3.1 for $d = 7, 11$. Proofs for $d = 1, 3$ are found in [Hur87] and a proof for $d = 2$ in [Lun36] (§VII, Satz 11). Monotonicity for $d = 7, 11$ was stated in [Lak73] without proof (for reasons obvious to anyone reading what follows). All of the proofs follow the same basic outline, with $d = 11$ being the most tedious.

Proof of Proposition 3.3.1. For the purposes of this proof, define $k_n = q_n/q_{n-1}$; we will show $|k_n| > 1$. We will also use the notation $B_t(r)$ for the ball of radius t centered at $r \in \mathcal{O}$. When $n = 1$ we have $|k_1| = |a_1| \geq 1/\rho > 1$. The recurrence $k_n = a_n + 1/k_{n-1}$ is immediate from the definitions. We argue by contradiction following [Hur87]. Suppose $n > 1$ is the smallest value for which $|k_n| \leq 1$ so that $|k_i| > 1$ for $1 \leq i < n$. Then

$$|a_n| = |k_n - 1/k_{n-1}| < 2.$$

For small values of a_i, those for which $(a_i + V) \cap V^{-1} \cap (\mathbb{C} \setminus V^{-1}) \neq \emptyset$, the values of a_{i+1} are restricted (cf. Figure 3.5). More generally, there are arbitrarily long "forbidden sequences" stemming from these small values of a_i. We will use some of the forbidden sequences that arise in this fashion to show that the assumption $k_n < 1$ leads to a contradiction.

- ($d = 7$) The only allowed values of a_n for which $|a_n| < 2$ are $a_n = \frac{\pm 1 \pm \sqrt{-7}}{2}$. By symmetry, we suppose $a_n = \frac{1+\sqrt{-7}}{2} =: \omega$ without loss of generality. It follows that $k_n \in B_1(\omega) \cap B_1(0)$. Subtracting $\omega = a_n$, we see that $1/k_{n-1} \in B_1(0) \cap B_1(-\omega)$. Applying $1/z$ then gives $k_{n-1} \in B_1(\omega - 1) \setminus B_1(0)$. Since $k_{n-1} = a_{n-1} + 1/k_{n-2}$, the only possible values for a_{n-1} are ω, $\omega - 1$, $\omega - 2$, $2\omega - 1$, and $2\omega - 2$. One can verify that the two-term sequences

a_i	a_{i+1}
$\omega - 2$	ω
$\omega - 1$	ω
ω	ω

are forbidden, so that $a_{n-1} = 2\omega - 1$ or $2\omega - 2$. We now have either $k_{n-1} \in B_1(2\omega - 1) \cap B_1(\omega - 1)$ if $a_{n-1} = 2\omega - 1$, or $k_{n-1} \in B_1(2\omega - 2) \cap B_1(\omega - 1)$ if $a_{n-1} = 2\omega - 2$. Subtracting a_{n-1} and applying $1/z$ shows that $a_{n-2} = 2\omega - 1$ or $2\omega - 2$ if $a_{n-1} = 2\omega - 1$, and $a_{n-2} = 2\omega$ or $2\omega - 1$ if $a_{n-1} = 2\omega - 2$.

Continuing shows that for $i \leq n - 1$

$$k_i \in (B_1(2\omega - 2) \cup B_1(2\omega - 1) \cup B_1(2\omega)) \cap (B_1(\omega - 1) \cup B_1(\omega)) \, ,$$

the green region on the left in Figure 3.7. This is impossible; for instance $k_1 = a_1 \in \mathcal{O}$ but the region above contains no integers.

- ($d = 11$) The only allowed values of a_n for which $|a_n| < 2$ are $\frac{\pm 1 \pm \sqrt{-11}}{2}$. By symmetry, we suppose $a_n = \frac{1+\sqrt{-11}}{2} =: \omega$ without loss of generality. Hence $k_n \in B_1(0) \cap B_1(\omega)$. Subtracting a_n and applying $1/z$ shows that $k_{n-1} \in B_{1/2}(\frac{\omega-1}{2}) \setminus B_1(0)$ and $a_{n-1} = \omega - 1$ or ω. If $a_{n-1} = \omega - 1$, subtracting, applying $1/z$ and using the three-term forbidden sequences

a_i	a_{i+1}	a_{i+2}
$\omega - 1$	$\omega - 1$	ω
ω	$\omega - 1$	ω
$\omega + 1$	$\omega - 1$	ω

shows that $a_{n-2} = 2\omega$ or $2\omega - 1$, with $k_{n-2} \in B_1(2\omega) \cap B_1(\omega)$ or $B_1(2\omega - 1) \cap B_1(\omega)$ respectively.

If $a_{n-1} = \omega$, subtracting and applying $1/z$ gives $a_{n-2} = \omega - 2$ or $\omega - 1$ with $k_{n-2} \in B_{1/2}(\frac{\omega-2}{2}) \cap B_1(\omega - 2)$ or $(B_{1/2}(\frac{\omega-2}{2}) \cap B_1(\omega - 1)) \setminus B_{1/2}(\frac{\omega-1}{2})$ respectively. If $a_{n-2} = \omega - 1$, subtracting a_{n-2}, applying $1/z$, and using the three-term forbidden sequences

a_i	a_{i+1}	a_{i+2}
$\omega - 1$	$\omega - 1$	ω
$\omega - 2$	$\omega - 1$	ω

shows that $a_{n-3} = 2\omega - 2$ or $2\omega - 1$ with $k_{n-3} \in B_1(2\omega-2) \cap B_1(\omega-1)$ or $B_1(2\omega-1) \cap B_1(\omega-1)$ respectively. If $a_{n-2} = \omega - 2$, subtracting a_{n-2} and applying $1/z$ shows that $a_{n-3} = \omega$ or $\omega + 1$, with $k_{n-3} \in (B_1(\omega) \cap B_{1/2}(\frac{\omega+1}{2})) \setminus B_1(0)$ or $B_1(\omega+1) \cap B_{1/2}(\frac{\omega+1}{2})$ respectively. If $a_{n-3} = \omega + 1$, we loop back into a symmetric version of a case previously considered (namely $k_{n-4} \in B_{1/2}(\frac{\omega-2}{2}) \setminus B_1(0)$ and $a_{n-4} = \omega - 1$ or $\omega - 2$). If $a_{n-3} = \omega$, subtracting a_{n-3}, applying $1/z$, and using the forbidden sequences

a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
$\omega - 1$	ω	$\omega - 2$		
ω	ω	$\omega - 2$	ω	
$\omega + 1$	ω	$\omega - 2$	ω	ω

shows that $a_{n-4} = 2\omega$ or $2\omega - 1$ with $k_{n-4} \in B_1(2\omega) \cap B_1(\omega)$ or $B_1(2\omega - 1) \cap B_1(\omega)$ resepectively.

Continuing, we find k_i for $i \leq n-1$ restricted to the region depicted on the right in Figure 3.7. This region contains no integers, contradicting $k_1 \in \mathcal{O}$.

□

3.4 An overview of A. L. Schmidt's continued fractions

Here we give a quick summary of Asmus Schmidt's continued fraction algorithm [Sch75a], its ergodic theory [Sch82], and further results of Hitoshi Nakada concerning these [Nak88a], [Nak90], [Nak88b]. Define the following matrices in $PGL(2, \mathbb{Z}[i])$:

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \; V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \; V_3 = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix},$$

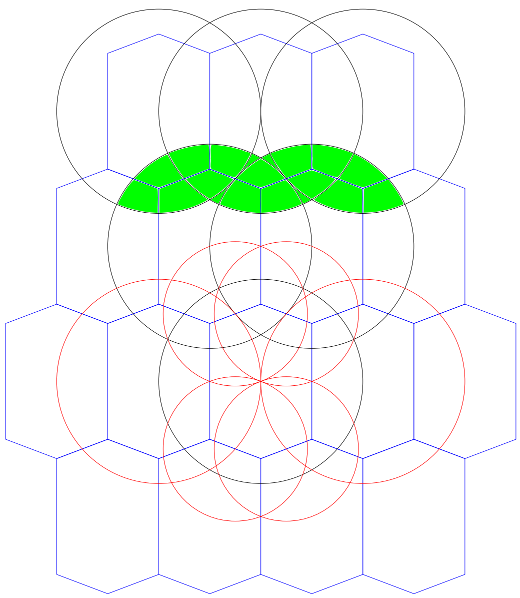

Figure 3.7: In the proof of Proposition 3.3.1, the assumption $k_n < 1$ with $a_n = \frac{1+\sqrt{-7}}{2}$ (left) or $a_n = \frac{1+\sqrt{-11}}{2}$ (right) leads to restricted potential values for k_i with $i < n$ (green regions).

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}, \ E_2 = \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}, \ E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}.$$

Note that

$$S^{-1}V_iS = V_{i+1}, \ S^{-1}E_iS = V_{i+1}, \ S^{-1}CS = C \text{ (indices modulo 3)}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ is elliptic of order three, and

$$\tau \circ m \circ \tau = m^{-1}$$

for the Möbius transformations m induced by $\{V_i, E_i, C\}$ (here τ is complex conjugation).

In [Sch75a] Schmidt uses infinite words in these letters to represent complex numbers as infinite products $z = \prod_n T_n, T_n \in \{V_i, E_i, C\}$ in two different ways. Let $M_N = \prod_{n=1}^N T_n$. We have **regular chains**

$$\det M_N = \pm 1 \Rightarrow T_{n+1} \in \{V_i, E_i, C\}, \ \det M_N = \pm i \Rightarrow T_{n+1} \in \{V_i, C\},$$

representing z in the upper half-plane $\mathcal{I}$ (the model circle) and **dually regular chains**

$$\det M_N = \pm i \Rightarrow T_{n+1} \in \{V_i, E_i, C\}, \ \det M_N = \pm 1 \Rightarrow T_{n+1} \in \{V_i, C\},$$

representing $z \in \{0 \leq x \leq 1, y \geq 0, |z - 1/2| \geq 1/2\} =: \mathcal{I}^*$ (the model triangle). The model circle is a disjoint union of four triangles and three circles, and the model triangle is a disjoint union of three triangles and one circle (see Figure 3.8):

$$\mathcal{I} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{C}$$

$$\mathcal{I}^* = \mathcal{V}_1^* \cup \mathcal{V}_2^* \cup \mathcal{V}_3^* \cup \mathcal{C}^*$$

where

$$\mathcal{V}_i = v_i(\mathcal{I}), \ \mathcal{E}_i = e_i(\mathcal{I}^*), \ \mathcal{C} = c(\mathcal{I}^*), \ \mathcal{V}_i^* = v_i(\mathcal{I}^*), \ \mathcal{C}^* = c(\mathcal{I}),$$

(lowercase letters indicating the Möbius transformation associated to the corresponding matrix).

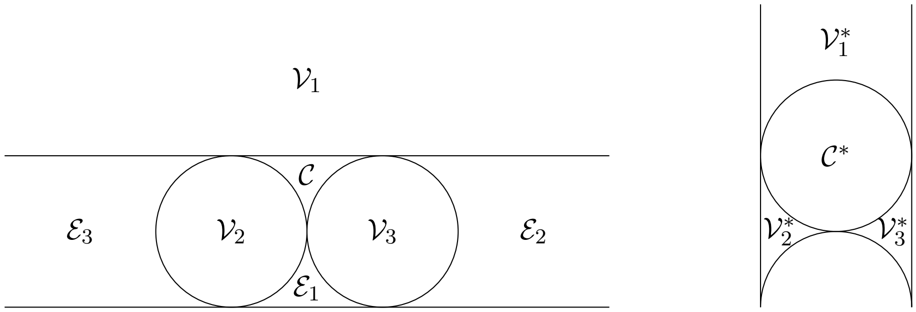

Figure 3.8: Subdivision of the upper half-plane $\mathcal{I}$ (model circle) and ideal triangle $\mathcal{I}^*$ with vertices $\{0, 1, \infty\}$ (model triangle). The continued fraction map T sends the circular regions onto $\mathcal{I}$ and the triangular regions onto $\mathcal{I}^*$.

By considering $z = \prod_n T_n$ we obtain rational approximations $p_i^{(N)}/q_i^{(N)}$ to z by

$$M_N \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} p_1^{(N)} & p_2^{(N)} & p_3^{(N)} \\ q_1^{(N)} & q_2^{(N)} & q_3^{(N)} \end{pmatrix}$$

(which are the orbits of $\infty, 0, 1$ under the partial products $m_N = t_1 \circ \cdots \circ t_N$).

The shift map T on $X = \mathcal{I} \cup \mathcal{I}^* = \{\text{chains, dual chains}\}$ maps X to itself via Möbius transformations, specifically (mapping $\mathcal{V}_i, \mathcal{C}^*$ onto $\mathcal{I}$ and $\mathcal{V}_i^*, \mathcal{E}_i, \mathcal{C}$ onto $\mathcal{I}^*$)

$$T(z) = \begin{cases} v_i^{-1} z & z \in \mathcal{V}_i \cup \mathcal{V}_i^* \\ e_i^{-1} z & z \in \mathcal{E}_i \\ c^{-1} z & z \in \mathcal{C} \cup \mathcal{C}^* \end{cases}.$$

The shift $T : X \to X$ is shown to be ergodic ([Sch82], Theorem 5.1) with respect to the following probability measure

$$\tilde{f}(z) = \begin{cases} \frac{1}{2\pi^2}(h(z) + h(sz) + h(s^2 z)) & z = x + yi \in \mathcal{I} \\ \frac{1}{2\pi}\frac{1}{y^2} & z = x + yi \in \mathcal{I}^* \end{cases}$$

where

$$h(z) = \frac{1}{xy} - \frac{1}{x^2} \arctan\left(\frac{x}{y}\right).$$

By inducing to $X \setminus \cup_i (\mathcal{V}_i \cup \mathcal{V}_i^*)$ Schmidt gives "faster" convergents $\hat{p}_\alpha^{(n)}/\hat{q}_\alpha^{(n)}$ and a sequence of exponents e_n (1 for E_i, C, and the return time k for V_i^k). He then gives results analagous to those of simple continued fractions via the pointwise ergodic theorem, including the arithmetic and geometric mean of the exponents which exist for almost every z ([Sch82] theorem 5.3)

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n} e_i\right)^{1/n} = 1.26\ldots,\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e_i = 1.6667\ldots.$$

In [Nak88a], Nakada constructs an invertible extension of T on a space of geodesics in two copies of three-dimensional hyperbolic space. In one copy we take geodesics from $\overline{\mathcal{I}^*}$ to $\mathcal{I}$ and in the other the geodesics from $\overline{\mathcal{I}}$ to $\mathcal{I}^*$ where the overline indicates complex conjugation. The regions are shown in Figure 3.9. The extension acts as Schmidt's T depending on the second coordinate. Nakada doesn't provide a second proof of ergodicity, but quotes Schmidt's result. Also in [Nak88a], results about the the density of Gaussian rationals p/q that appear as convergents and satisfy $|z - p/q| < c/|q|^2$ are obtained. For instance ([Nak88a], theorem 7.3), for almost every $z \in X$ and $0 < c < \frac{1}{1+1\sqrt{2}}$ it holds that

$$\lim_{N\to\infty}\frac{1}{N}\#\{p/q \in \mathbb{Q}(i) : p/q = p_i^{(n)}/q_i^{(n)},\ 1 \le n \le N,\ i = 1,2,3,\ |z - p/q| < c/|q|^2\} = \frac{c^2}{\pi}.$$

In [Nak90], Main Theorem, Nakada describes the rate of convergence of Schmidt's convergents. Namely for almost every z

$$\lim_{n\to\infty}\frac{1}{n}\log|q_i^{(n)}| = \frac{E}{\pi},\ \lim_{n\to\infty}\frac{1}{n}\log\left|z - \frac{p_i^{(n)}}{q_i^{(n)}}\right| = -\frac{2E}{\pi},\ E = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

For reference, the relationship between the Super-Apollonian Möbius generators $\{\mathfrak{s}_i, \mathfrak{s}_i^\perp\}$ (see chapter 4 for definitions) and Schmidt's $\{v_i, e_i, c\}$ are

$$\mathfrak{s}_1 = c^2 \circ \tau,\ \mathfrak{s}_2 = e_1^2 \circ \tau,\ \mathfrak{s}_3 = e_2^2 \circ \tau,\ \mathfrak{s}_4 = e_3^2 \circ \tau,$$

$$\mathfrak{s}_1^\perp = 1 \circ \tau,\ \mathfrak{s}_2^\perp = v_1^2 \circ \tau,\ \mathfrak{s}_3^\perp = v_2^2 \circ \tau,\ \mathfrak{s}_4^\perp = v_3^2 \circ \tau.$$

Schmidt gave similar algorithms over $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-11})$ in [Sch11] and [Sch78].

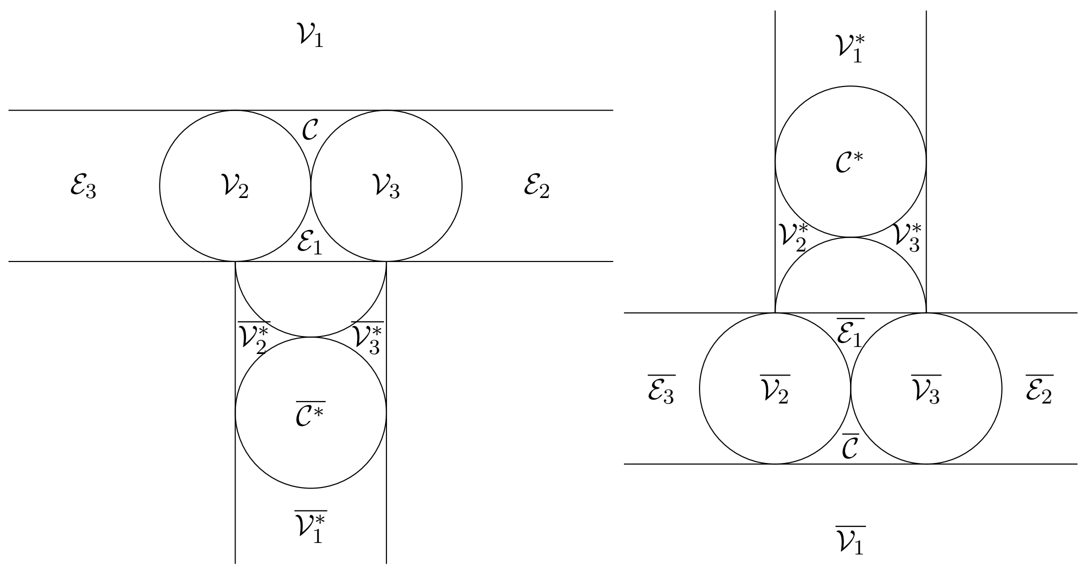

Figure 3.9: Domain of the invertible extension, $\mathcal{I} \cup \overline{\mathcal{I}^*}$ (left) and $\mathcal{I}^* \cup \overline{\mathcal{I}}$ (right).

Chapter 4

Continued fractions for some right-angled hyperbolic Coxeter groups

4.1 Introduction

In this chapter we discuss approximation algorithms in the plane based on circle packings coming from ideal, right-angled, two-colorable polyhedra in $\mathcal{H}^3$ (i.e. hyperbolic polyhedra all of whose vertices lie on $\partial\mathcal{H}^3$, whose faces are two-colorable with respect to incidence at edges, and whose faces meet only at right angles). After this general construction, we discuss the simpler example of billiards in the ideal hyperbolic triangle (restriction to the real line of the octahedral and cubeoctahedral examples of 4.4 and 4.5) and some of its properties. We then discuss two examples in some detail (octahedral and cubeoctrahedral continued fractions), with applications to Diophantine approximation over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$. These algorithms are variations on those of A. L. Schmidt [Sch75a], [Sch11], although precisely relating them is difficult (e.g. there doesn't seem to be a simple intertwining map). Note that the discussion of "Super-Apollonian" continued fractions has substantial overlap with our [CFHS17].

4.2 A general construction

4.2.1 Right-angled hyperbolic Coxeter groups from two-colorable polyhedra

Let Π be a polyhedron whose faces can be two-colored, i.e. the faces of Π can be partitioned into two sets $S = \{s_i\}$ and $T = \{t_j\}$ such that no two faces in S share an edge and no two faces in

T share an edge. We consider right-angled hyperbolic Coxeter groups of the form

$$\Gamma = \langle s_1, \ldots, s_m, t_1, \ldots, t_n | s_i^2 = t_i^2 = [s_i, t_j] = 1, s_i \sim t_j \rangle,$$

where $s_i \sim t_j$ if the faces s_i and t_j share an edge. In other words, we are considering reflections in the sides of an ideal (all vertices at infinity) right-angled hyperbolic polyhedron whose faces can be two-colored. Every combinatorial type of polyhedron whose faces can be two-colored is realizable as an ideal right-angled hyperbolic polyhedron. This is a consequence of the so-called Koebe-Andreev-Thurston theorem, that every connected simple planar graph can be realized as the tangency graph of disks in the plane. See [KN17] for a discussion in this context.

The ideal boundaries of the planes defining the faces of Π consist of oriented circles $Z(s_i)$, $Z(t_j)$ with the property that the interiors of $Z(s)$, $s \in S$ are disjoint, the interiors of $Z(t)$, $t \in T$ are disjoint, together they cover the sphere, and if $Z(s)$ and $Z(t)$ intersect, they do so at right angles.

4.2.2 A pair of dynamical systems

Denote the interiors of $Z(s)$, $Z(t)$, by $C(s)$, $C(t)$. The sets

$$P(t) := \overline{C(t)} \cap \left(S^2 \setminus \cup_{s\in S} C(s)\right),\ P(s) = \overline{C(s)} \cap \left(S^2 \setminus \cup_{t\in T} C(t)\right)$$

have polygonal boundary with vertices on $Z(t)$ and $Z(s)$ respectively (the overline indicates closure, so as to include the vertices in the closed sets $P(s)$, $P(t)$). We have two partitions (excepting shared vertices of the polygonal faces P)

$$S^2 = (\cup_{s\in S} C(s)) \cup (\cup_{t\in T} P(t)) = (\cup_{s\in S} P(s)) \cup (\cup_{t\in T} C(t))$$

of the sphere, the first $\mathcal{P}_S$ making the faces in S open disks and the faces in T closed polygons, and the second $\mathcal{P}_T$ making the faces in T open disks and the faces in S closed polygons. We think of $\mathcal{P}_S$ and $\mathcal{P}_T$ "preferring" S and T respectively. See Figures 4.18 and 4.19 for an example.

We will use these partitions to define a pair of dynamical systems, "dual" in the sense that their invertible extensions are inverses to one another. Let $s_i, t_j \in PGL_2(\mathbb{C}) \rtimes \langle \mathfrak{c} \rangle \cong \mathrm{Isom}(H^3)$, where $\mathfrak{c}$ is complex conjugation, be a realization of Γ as a discrete group of hyperbolic isometries in the upper half-space model. Define two dynamical systems on $S^2 = P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ as follows

$$\phi_S(z) = \begin{cases} s(z) & z \in C(s) \\ t(z) & z \in P(t) \end{cases}, \ \phi_T(w) = \begin{cases} s(w) & w \in P(s) \\ t(w) & w \in C(t) \end{cases}.$$

(Note that ϕ_S, ϕ_T are well-defined at the vertices, which are fixed points.)

The invertible extensions Φ_S and Φ_T are defined on a collection of geodesics (pairs of distinct points of $P^1(\mathbb{C})$) as follows. For each $s \in S$ and each $t \in T$, define the dual regions

$$C^*(s) = \left(\cup_{t'} C(t')\right) \cup \left(\cup_{t'} P(t')\right), \ P^*(s) = \left(\cup_{t'} C(t')\right) \cup \left(\cup_{t'} P(t')\right),$$

$$C^*(t) = \left(\cup_{s'} C(s')\right) \cup \left(\cup_{s'} P(s')\right), \ P^*(t) = \left(\cup_{s'} C(s')\right) \cup \left(\cup_{s'} P(s')\right),$$

where the primed index in the definition of $C^*(s)$ is over all $t \in T$ such that the regions $C(t')$, $P(t')$, do not intersect $C(s)$ (and similarly for the three other dual regions). The invertible extensions are defined on the pairs (thought of as geodesics going from w in one of the dual regions to z in one of the circular or polygonal regions for Φ_S and *vice versa* for Φ_T)

$$\begin{aligned}(w, z) \in \mathcal{G} &= (\cup_{s \in S} C^*(s) \times C(s)) \cup (\cup_{t \in T} P^*(t) \times P(t)) \\ &= (\cup_{s \in S} P^*(s) \times P(s)) \cup (\cup_{t \in T} C^*(t) \times C(t)).\end{aligned}$$

On $\mathcal{G}$, we have (acting diagonally, extending ϕ_S in the second coordinate and ϕ_T in the first coordinate respectively)

$$\Phi_S(w, z) = (r(w), \phi_S(z)) \ \text{ where } \ \phi_S(z) = r(z), \ r \in S \cup T,$$

$$\Phi_T(w, z) = (\phi_T(w), q(z)) \ \text{ where } \ \phi_T(w) = q(w), \ q \in S \cup T.$$

The reader can convince themselves that the Φ are bijections on the stated domains (see Figure 4.16 for an example of the domain for the invertible extension in the octahedral case). Moreover, the extensions Φ_S and Φ_T are inverse to one another.

4.2.3 Symbolic dynamics

The maps ϕ and Φ are one- and two-sided subshifts of finite type on the alphabet $S \cup T$. The sequences in question come from the two "obvious" normal forms for elements of Γ. If we consider ϕ_S, then the sequence

$$(\phi_S(z), \phi_S^2(z), \phi_S^3(z), \ldots) = (r_1(z), r_2r_1(z), r_3r_2r_1(z), \ldots),\ r_n \in S \cup T,$$

has the following properties.

- $r_n \neq r_{n+1}$, i.e. no words of the form s^2 or t^2 appear (inversions in the boundary circles of $C(s)$ or $P(t)$ ensure you do not repeat a digit).

- If r_n and r_{n+1} commute (i.e. their fixed circles are orthogonal), then $r_n \in S$ and $r_{n+1} \in T$. Geometrically, this comes from the fact that we are taking $C(s)$ and $P(t)$ in the definition, i.e. we prefer S over T in the definition.

Similarly, the map ϕ_T will produce a sequence

$$(\phi_T(w), \phi_T^2(w), \phi_T^3(w), \ldots) = (q_1(w), q_2q_1(w), q_3q_2q_1(w), \ldots),\ q_n \in S \cup T,$$

satisfying the opposite convention for the commutation relations: if q_n and q_{n+1} commute, then $q_n \in T$ and $q_{n+1} \in S$.

Hence, to a point $w \in \mathbb{C}$ or $z \in \mathbb{C}$, we encode the pair (w, z) as

$$(\ldots q_3q_2q_1, r_1r_2r_3\ldots),$$

and one sees that Φ_S and Φ_T are the left- and right-shift on the bi-infinite word (concatenating the q- and r-strings), and that this bi-infinite word (read left to right) is in the normal form described above for the r-sequence (i.e. s before t when they commute and no instances of s^2 or t^2).

4.2.4 Invariant measures and ergodicity

The maps Φ are bijections defined piecewise by isometries, and therefore preserve the isometry invariant measure $d\eta = |z-w|^{-4}dudvdxdy$ (where $w = u+iv$, $z = x+iy$) on the space of geodesics

$\mathcal{G}$. [This is simple to verify or derive as was done for $\mathcal{H}^2$ in 3.2.1] This pushforward of this measure onto the first or second coordinate gives explicit invariant measures for the continued fraction maps ϕ. See 4.4.6 and 4.5.1 for examples.

The measure preserving systems described in this chapter are almost certainly ergodic, but I have not been able to prove it. The frame flow and geodesic flow on the finite volume quotient $\Gamma\backslash\mathcal{H}^3$ are ergodic, but the systems described in this section seem to live in some hybrid of cross-sections of flows on the two infinite volume quotients $\langle s \in S\rangle\backslash\mathcal{H}^3$ and $\langle t \in T\rangle\backslash\mathcal{H}^3$. A. L. Schmidt gives a direct argument for ergodicity of his continued fraction map in [Sch82].

4.3 Billiards in the ideal triangle

Consider the group $\Gamma = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rangle \subseteq PGL_2(\mathbb{R}) \cong \mathrm{Isom}(\mathcal{H}^2)$, generated by reflections in the walls of the ideal hyperbolic triangle with vertices $\{0, 1, \infty\}$:

$$\mathfrak{a}(x) = -x,\ \mathfrak{b}(x) = \frac{x}{2x-1},\ \mathfrak{c}(x) = 2-x.$$

We have the short exact sequence and isomorphisms

$$1 \to \Gamma \to PGL_2(\mathbb{Z}) \to PGL_2(\mathbb{Z}/(2)) \to 1,$$

$$PGL_2(\mathbb{Z}) = \Gamma \rtimes S_3,\ \Gamma \cong \mathbb{Z}/(2) * \mathbb{Z}/(2) * \mathbb{Z}/(2).$$

Reduced words in these generators index the triangles in the associated tessellation, and the words of length n partition the line into $3 \cdot 2^{n-1}$ subintervals. The partition by words of length $m > n$ refines the partition by words of length n. Irrational x are then uniquely coordinatized by infinite words in the alphabet $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$. See Figure 4.1.

The expansion of x as an infinite word in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is produced by a dynamical system $T : P^1(\mathbb{R}) \to P^1(\mathbb{R})$:

$$T(x) = \begin{cases} \mathfrak{a}(x) = -x & x \in [-\infty, 0] \\ \mathfrak{b}(x) = \frac{x}{2x-1} & x \in [0, 1] \\ \mathfrak{c}(x) = 2-x & x \in [1, \infty]. \end{cases}$$

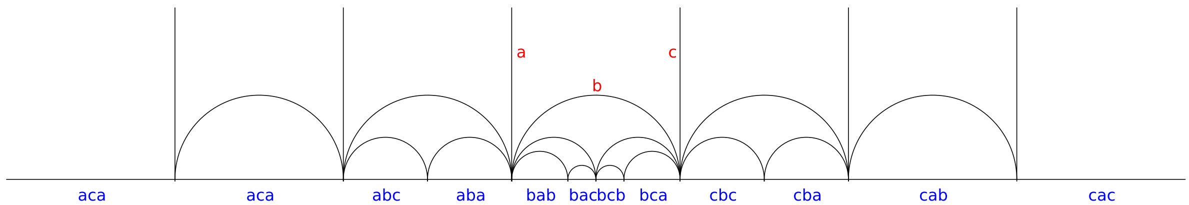

Figure 4.1: Partion of the line induced by reduced words of length three in the Coxeter generators.

There are three neutral fixed points, $\frac{0}{1}$, $\frac{1}{1}$, and $\frac{1}{0}$, to which rational points descend in finitely many steps depending on the parity of the numerator and denominator (even/odd, odd/odd, odd/even).

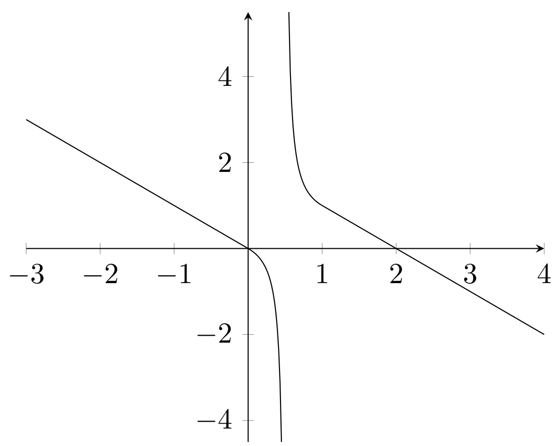

Figure 4.2: Graph of $T(x)$, with fixed points at 0, 1, and ∞.

If x is irrational and $\mathfrak{m} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is defined by $T^n x = \mathfrak{m}(T^{n-1}x)$, then we obtain three sequences of rational convergents by following the inverse orbit of $\{0, 1, \infty\}$,

$$\lim_{n\to\infty} \mathfrak{m}_1\mathfrak{m}_2 \dots \mathfrak{m}_n x_0 = x, \; x_0 = 0, 1, \infty,$$

i.e. we keep track of the the vertices of the triangles through which the geodesic $\overrightarrow{\infty x}$ passes. This algorithm is a slow, reflective, version of the usual continued fraction algorithm. The convergents can be constructed starting from $(1/1, \pm 1/0, 0/1)$, according as x is positive or negative, and taking mediants, moving through the Farey tree. Applying $\mathfrak{a}$, $\mathfrak{b}$, or $\mathfrak{c}$ updates the first, second, or third position by taking the mediant $\frac{p}{q} \oplus \frac{r}{s} := \frac{p+q}{r+s}$ of the other two entries.

For example, the first 20 convergents of the random number

$$x = 0.4189513796210592\ldots,\ \mathfrak{x} = \mathfrak{bacabcacbcacacababac}\ldots,$$

are (mediants in italics, see Figure 4.3):

$(1/1, 1/0, 0/1)$, $(1/1, \mathit{1/2}, 0/1)$, $(\mathit{1/3}, 1/2, 0/1)$, $(1/3, 1/2, \mathit{2/5})$, $(\mathit{3/7}, 1/2, 2/5)$, $(3/7, \mathit{5/12}, 2/5)$, $(3/7, 5/12, \mathit{8/19})$, $(\mathit{13/31}, 5/12, 8/19)$, $(13/31, 5/12, \mathit{18/43})$, $(13/31, \mathit{31/74}, 18/43)$, $(13/31, 31/74, \mathit{44/105})$, $(\mathit{75/179}, 31/74, 44/105)$, $(75/179, 31/74, \mathit{106/253})$, $(\mathit{137/327}, 31/74, 106/253)$, $(137/327, 31/74, \mathit{168/401})$, $(\mathit{199/475}, 31/74, 168/401)$, $(199/475, \mathit{367/876}, 168/401)$, $(\mathit{535/1277}, 367/876, 168/401)$, $(535/1277, \mathit{703/1678}, 168/401)$, $(\mathit{871/2079}, 703/1678, 168/401)$, $(871/2079, 703/1678, \mathit{1574/3757})$

$= (0.4189514\ldots, 0.4189511\ldots, 0.4189512\ldots)$.

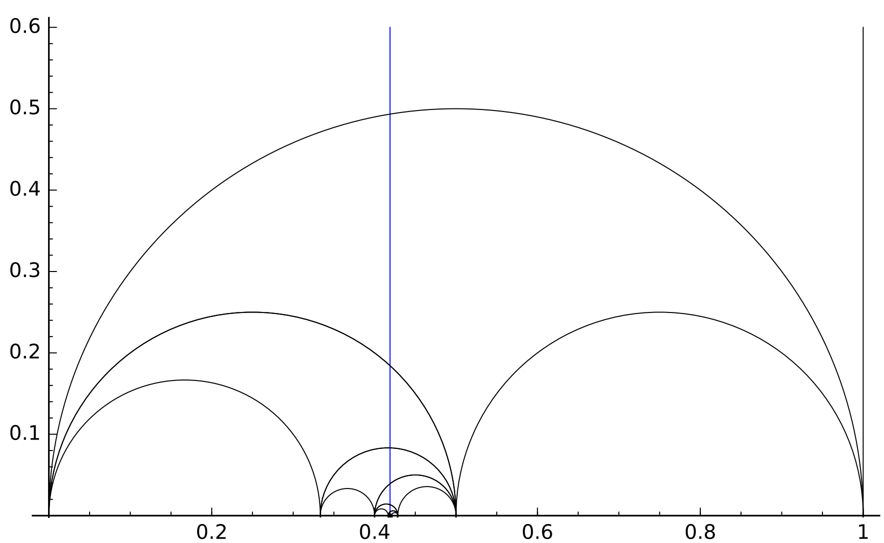

Figure 4.3: Approximating $x = 0.4189513796210592\ldots$, $\mathfrak{x} = \mathfrak{bacabcacbcacacababac}\ldots$.

The invertible extension $\widetilde{T}$ of T is defined on the space of geodesics $\mathcal{G}$ that intersect the fundamental triangle, acting on $\overrightarrow{yx}$ depending on x:

$$\widetilde{T}(y,x) = \begin{cases} (\mathfrak{a}(y), \mathfrak{a}(x)) & x \in [-\infty, 0] \\ (\mathfrak{b}(y), \mathfrak{b}(x)) & x \in [0,1] \\ (\mathfrak{c}(y), \mathfrak{c}(x)) & x \in [1,\infty]. \end{cases}$$

$\widetilde{T}$ associates to the geodesic $\overrightarrow{yx}$ a bi-infinite word $\mathfrak{y}^{-1}\mathfrak{x}$ in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, which we will relate to the geodesic flow in $\Gamma\backslash\mathcal{H}^2$.

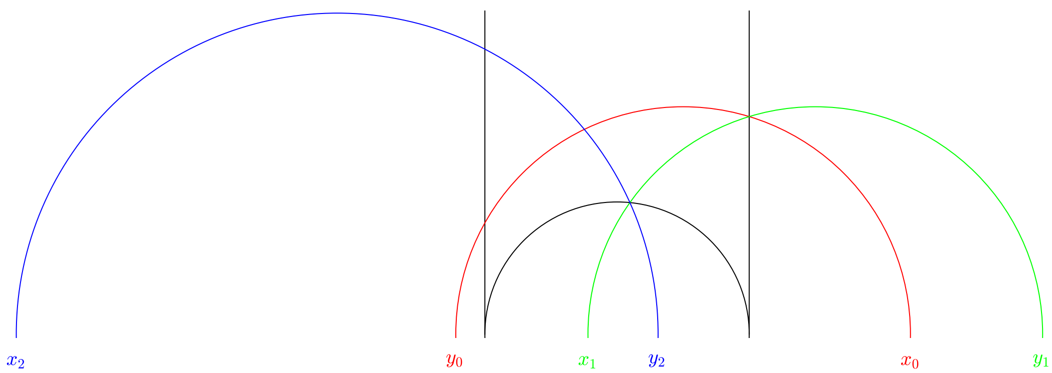

Figure 4.4: Two iterations of $\widetilde{T}$, red to green to blue

The measure $d\eta(y,x) = (x-y)^{-2}dxdy$ is isometry-invariant on the space of geodesics in the hyperbolic plane. Since $\widetilde{T}$ is a bijection defined piecewise by isometries, $\eta|_{\mathcal{G}}$ is $\widetilde{T}$-invariant. Pushing forward to the second coordinate gives an *infinite* T-invariant measure

$$d\mu(x) = \begin{cases} \frac{dx}{-x} & x < 0, \\ \frac{dx}{x(1-x)} & 0 < x < 1, \\ \frac{dx}{x-1} & x > 1, \end{cases}$$

We will see that $(\mathcal{G}, \widetilde{T}, \eta|_{\mathcal{G}})$ is ergodic (hence also the ergodicity of $(P^1(\mathbb{R}), T, \mu)$).

The word $\mathfrak{y}^{-1}\mathfrak{x}$ associated to the geodesic $\overrightarrow{yx}$ records the sequence of collisions with the walls of the triangle in $\Gamma\backslash\mathcal{H}^2$, and $\widetilde{T}$ is the first-return of the geodesic flow (billiards in the triangle) to the cross-section defined by points/directions on the boundary. The return time is integrable with respect to $d\eta(y,x)$. For instance, a geodesic $(y,x) \in [-\infty, 0] \times [1,\infty]$, has return time

$$r(y,x) = \frac{1}{2}\log\left(\frac{x(1-y)}{y(1-x)}\right).$$

and the integral is

$$\frac{1}{2}\int_{-\infty}^{0}\int_{1}^{\infty}\log\left(\frac{x(1-y)}{y(1-x)}\right)\frac{dxdy}{(y-x)^2}=\frac{\pi^2}{6}.$$

It is well-known (since E. Hopf [Hop71]) that the geodesic flow on finite volume hyperbolic surfaces is ergodic, from which it follows that $\widetilde{T}$ and T are ergodic.

4.4 Octahedral ("Super-Apollonian") continued fractions over $\mathbb{Q}(\sqrt{-1})$

[For the convenience of the author, we follow the notation of [CFHS17] which contains more on our "Super-Apollonian" continued fractions and its relationship with Apollonian circle packings and the Super-Apollonian group of [GLM+05], [GLM+06a], [GLM+06b].] In this section we describe the right-angled continued fractions generated by reflections in the sides of the ideal regular right-angled octahedron with vertices

$$0,\ 1,\ \infty,\ i,\ 1+i,\ \frac{1}{1-i}.$$

This reflection group Γ is generated by

$$\mathfrak{s}_1=\frac{(1+2i)\bar{z}-2}{2\bar{z}-1+2i},\ \mathfrak{s}_2=\frac{\bar{z}}{2\bar{z}-1},\ \mathfrak{s}_3=-\bar{z}+2,\ \mathfrak{s}_4=-\bar{z},$$

$$\mathfrak{s}_1^{\perp}=\bar{z},\ \mathfrak{s}_2^{\perp}=\bar{z}+2i,\ \mathfrak{s}_3^{\perp}=\frac{\bar{z}}{-2i\bar{z}+1},\ \mathfrak{s}_4^{\perp}=\frac{(1-2i)\bar{z}+2i}{-2i\bar{z}+1+2i}$$

and fits into the short exact sequence

$$1\to\Gamma\to PGL_2(\mathbb{Z}[i])\rtimes\langle\bar{z}\rangle\to PGL_2(\mathbb{Z}[i]/(2))\to 1,$$

with

$$[PGL_2(\mathbb{Z}[i])\rtimes\langle\bar{z}\rangle:\Gamma]=48,\ PGL_2(\mathbb{Z}[i])\rtimes\langle\bar{z}\rangle=\Gamma\rtimes\text{Bin. Oct.}.$$

4.4.1 Digression on the Apollonian super-packing and the Super-Apollonian group

While this subsection isn't necessary for understanding the rest of the chapter, it gives convenient language to discuss the continued fraction algorithms we present in the sequel since the partitions of the approximation algorithm come from circle packings.

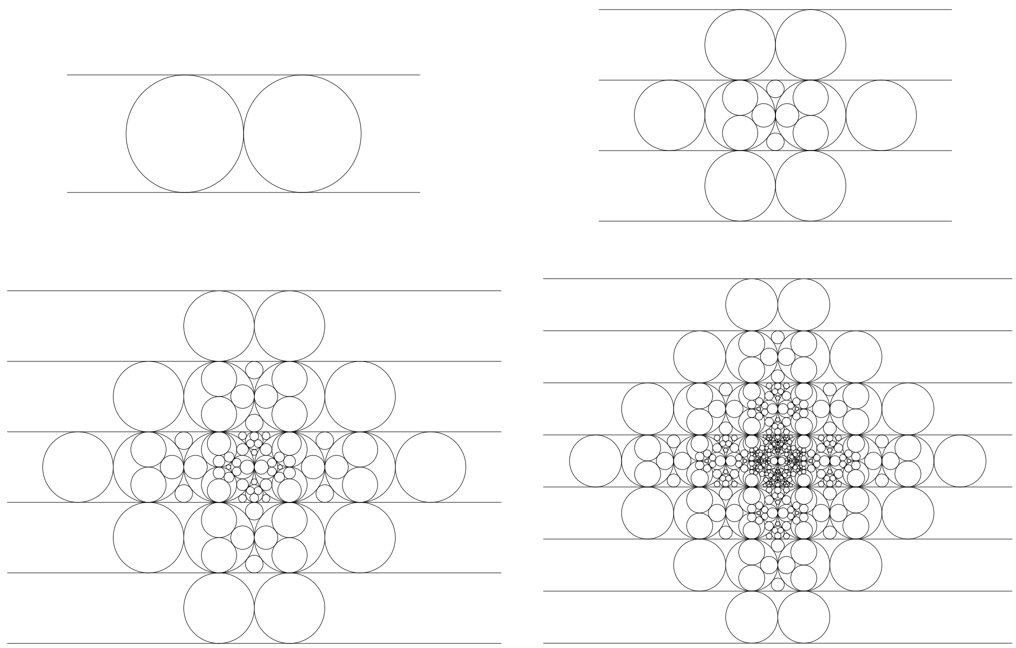

Figure 4.5: Partion of the plane induced by words of length 1, 2, 3, and 4 in the Coxeter generators (associated to the map T_B), or the orbit of the action of words from $\mathcal{A}^S$ of length 0, 1, 2, and 3 on the base quadruple R_B.

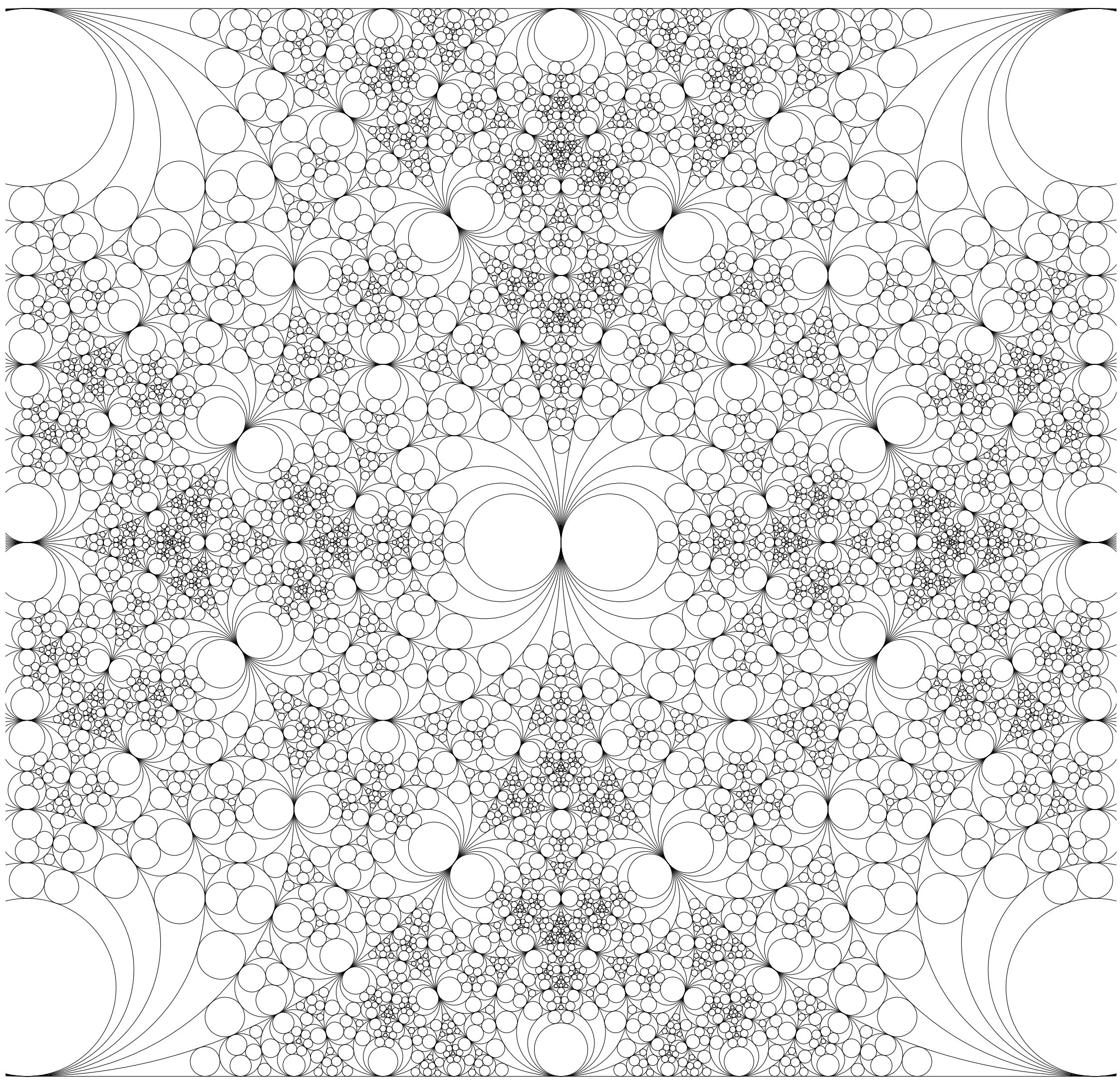

Figure 4.6: Circles in $[0,1] \times [0,1]$ in the orbit of words of length ≤ 5 from $\mathcal{A}^S$ acting on the base quadruple R_B.

A *Descartes quadruple* is a collection of four mutually tangent circles in the plane (ordered and oriented so that the interiors do not overlap), and its dual quadruple consists of the four mutually tangent circles passing orthogonally through its points of tangency. The curvatures (oriented inverse radii) of the circles satisfy

$$2(c_1^2 + c_2^2 + c_3^2 + c_4^2) = (c_1 + c_2 + c_3 + c_4)^2,$$

i.e. they are zeros of the quadratic form

$$D(x) = x \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} x^t.$$

We can move between "adjacent" quadruples with four *swaps* (fix three circles and replace the fourth by its inversion in the disjoint dual circle) and four *inversions* (fix one circle and replace the other three with their inversions in the fixed circle). See Figure 4.7 for an example of these operations.

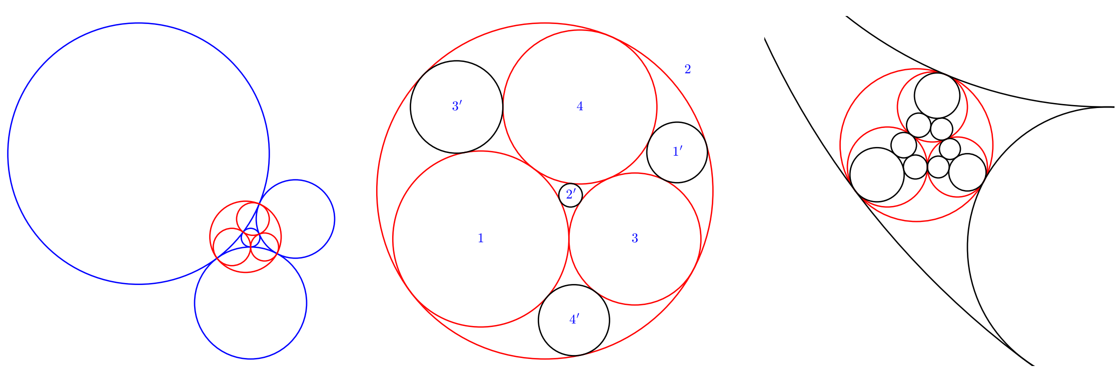

Figure 4.7: A Descartes quadruple and its dual, the four "swaps" S_i, and the four "inversions" $S_i^\perp$.

The *super-Apollonian group* $\mathcal{A}^S \subseteq O_D(\mathbb{Z}) \subseteq GL_4(\mathbb{Z})$ is the group generated by the swaps and inversions ($S_i^\perp := S_i^t$)

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

$\mathcal{A}^S$ is a **right-angled** hyperbolic Coxeter group with presentation

$$\langle S_i, S_i^\perp,\ 1 \le i \le 4 : S_i^2 = (S_i^\perp)^2 = [S_i, S_j^\perp] = 1,\ i \ne j \rangle.$$

The Super-Apollonian group acts on Descartes quadruples where the four circles are coordinatized by indefinite binary Hermitian forms of determinant -1 (called augmented curvature center or ACC coordinates in [GLM+06a]). To the circle

$$0 = A|z|^2 + B\bar{z} + \overline{B}z + C,\ A, C \in \mathbb{R},\ B \in \mathbb{C},\ AC - |B|^2 = -1$$

we associate the quadruple of real numbers

$$(C, A, B_1, B_2),$$

the co-curvature, curvature, x-coordinate of the curvature-center, and y-coordinate of the curvature-center respectively. Four circles (given by the columns of the 4×4 matrix R) form a Descartes quadruple if and only if they satisfy

$$R^t D R = \begin{pmatrix} 0 & -4 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The Super-Apollonian group acts by left-multiplication on such R as swaps and inversions. We choose two convenient "base quadruples," shown in Figure 4.8, for later use:

$$R_B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}, \ R_A = R_B^\perp = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

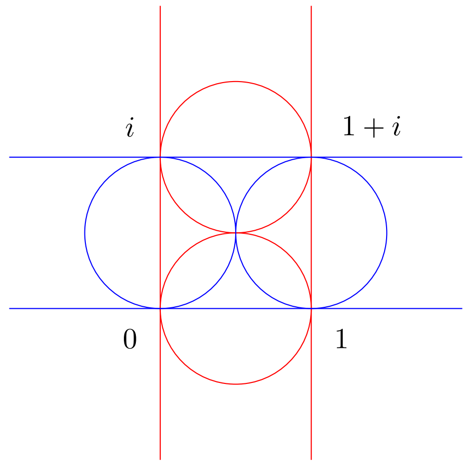

Figure 4.8: The dual base quadruples R_A (red) and R_B (blue).

4.4.2 A pair of dynamical systems

The pair of dynamical systems on $P^1(\mathbb{C})$ associated to the two-colorable octahedron with faces defined by $S := \{\mathfrak{s}_i\}$ and $S^\perp := \{\mathfrak{s}_i^\perp\}$ are as follows. Let B_i, B_i', be the open circular and closed triangular regions of the plane coming from $S^\perp$ (A_i and A_i' defined similarly, see Figure 4.9), and define

$$T_B(z) = \begin{cases} \mathfrak{s}_i z & z \in B_i', \\ \mathfrak{s}_i^\perp z & z \in B_i, \end{cases}, \; T_A(w) = \begin{cases} \mathfrak{s}_i w & w \in A_i, \\ \mathfrak{s}_i^\perp w & w \in A_i'. \end{cases}$$

Each of T_A and T_B has six fixed points, the points of tangency $\{0, 1, \infty, i, i+1, 1/(1-i)\}$.

Iteration of the map T_B with input z produces a word $\mathfrak{z} = \mathfrak{m}_1 \cdots \mathfrak{m}_n \cdots$ in the Möbius generators $S \cup S^\perp$, where $\mathfrak{m}_n$ is defined by $T_B^n z = \mathfrak{m}_n(T_B^{n-1}z)$. (Similarly for T_A.) We take the word to be finite for $z \in \mathbb{Q}(i)$, ending when a fixed point is reached (cf. 4.4.3). As discussed in 4.2.3, the words produced will be in "normal form," i.e.

- the words $(\mathfrak{s}_i^\perp)^2$, $\mathfrak{s}_i^2$ do not appear, and
- if $\mathfrak{m}_n$ and $\mathfrak{m}_{n+1}$ commute, then $\mathfrak{m}_n \in S^\perp$ and $\mathfrak{m}_{n+1} \in S$ (for T_B) or $\mathfrak{m}_n \in S$ and $\mathfrak{m}_{n+1} \in S^\perp$ (for T_A).

The collection of length n words in normal form partitions the plane into a collection of $9 \cdot 5^{n-1} - 1$ triangles and circles, which we call Farey circles and triangles following Schmidt [Sch75a].

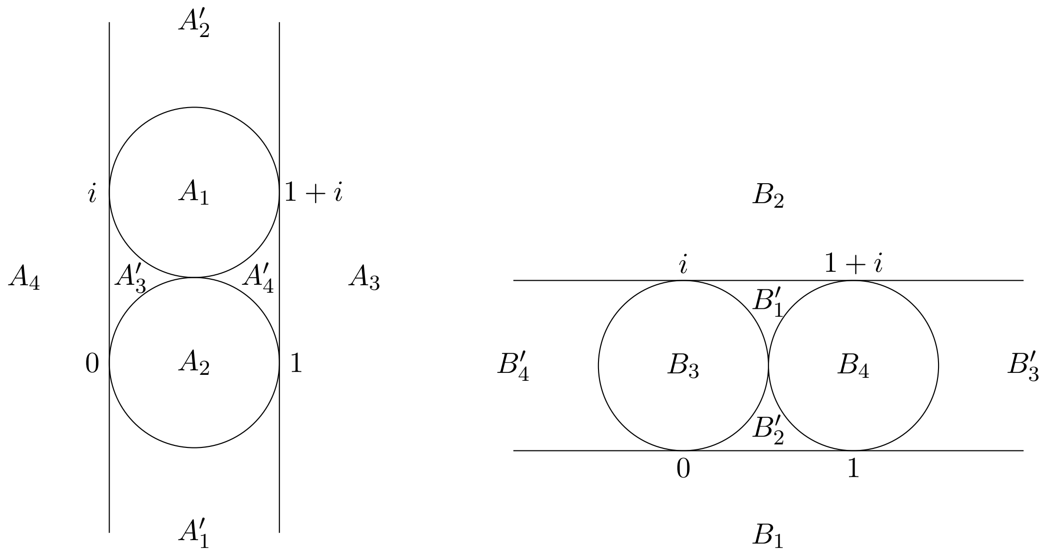

Figure 4.9: The regions A_i, A'_i, B_i, B'_i.

Specifically, we associate to each word an open circular or closed triangular region, the notation being

$$F_B(\mathfrak{m}) = \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_i \text{ (circular)}, \ \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_i' \text{ (triangular)},$$

for $\mathfrak{m} = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ with $\mathfrak{m}_n = \mathfrak{s}_i^\perp$ (circular) or $\mathfrak{s}_i$ (triangular). This definition is set up so that the words of length one correspond to the eight regions of the base quadruple. See Figure 4.10 for the partition and labels on words of length two and Figure 4.5 for the first four partitions.

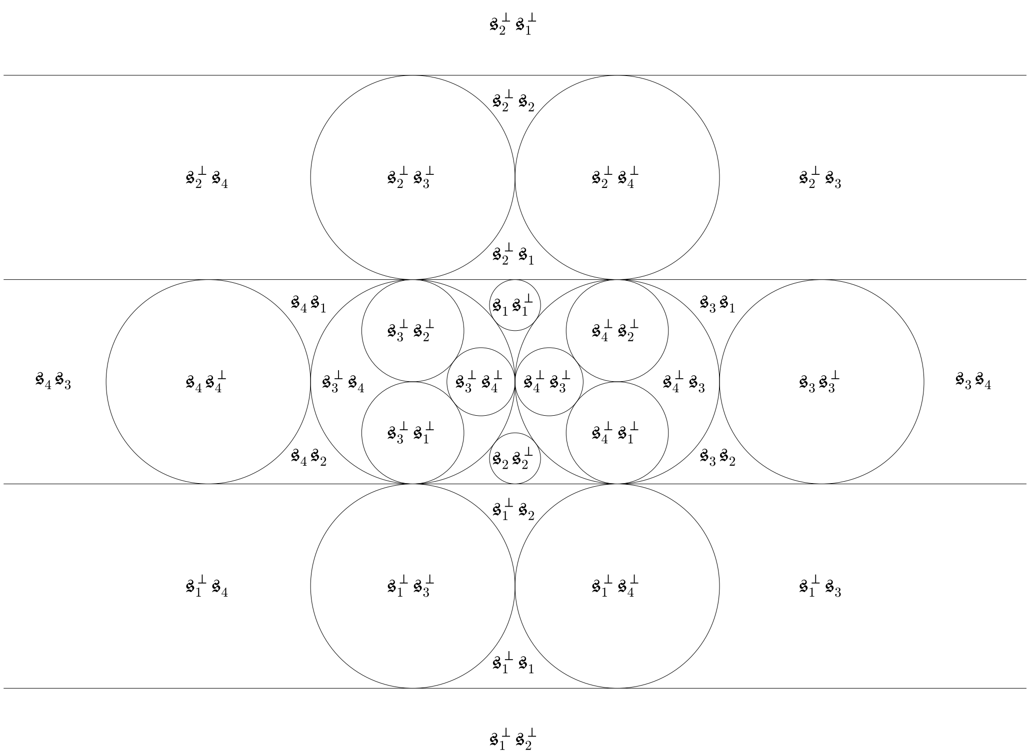

Figure 4.10: Partition of the complex plane induced by words of length two in swap normal form in the Möbius generators.

If z is rational, then $z = \mathfrak{z}v$ for a unique $v \in \{0, 1, \infty, i, 1+i, \frac{1}{1-i}\}$ matching z in parity as described in Theorem 4.4.1. If z is not rational, then $\mathfrak{z}$ is an infinite word with the property that

$$z = \bigcap_{n \geq 1} F(\mathfrak{m}_1 \cdots \mathfrak{m}_n).$$

This gives a bijection $z \leftrightarrow \mathfrak{z}$, under which T_B is the left shift operator, $T_B\left(\mathfrak{m}_1\mathfrak{m}_2\cdots\right) = \mathfrak{m}_2\mathfrak{m}_3\cdots$.

Figure 4.11 gives an example of the approximation afforded by this algorithm, "zooming in" on a particular z. The above discussion can be repeated *mutatis mutandi* for T_A.

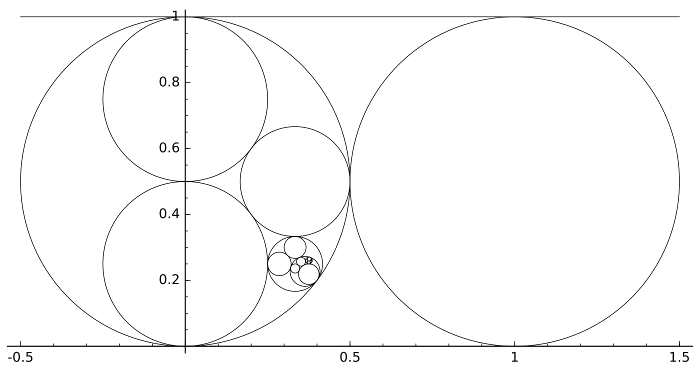

Figure 4.11: Approximating the irrational point $0.3828008104\ldots + i0.2638108161\ldots$ using T_B gives the normal form word $\mathfrak{s}_3^{\perp}\mathfrak{s}_2\mathfrak{s}_2^{\perp}\mathfrak{s}_3^{\perp}\mathfrak{s}_1\mathfrak{s}_1^{\perp}\mathfrak{s}_4\mathfrak{s}_2\mathfrak{s}_4\mathfrak{s}_1\mathfrak{s}_1^{\perp}\mathfrak{s}_3\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_3^{\perp}\mathfrak{s}_4\mathfrak{s}_4^{\perp}\mathfrak{s}_2\mathfrak{s}_1\mathfrak{s}_2\cdots$.

Before discussing the invertible extension(s) and invariant measure(s), we consider some arithmetic properties in the next few subsections.

4.4.3 A Euclidean algorithm

When applied to rational points p/q, iterating T_A and T_B performs a "slow, reflective" Euclidean algorithm, reducing the height of p/q until one of the vertices of the fundamental octahedron is reached.

Theorem 4.4.1. *Under the dynamical systems T_A and T_B, every Gaussian rational $z \in \mathbb{Q}(i)$ reaches one of the fixed points in finite time. The fixed point reached is determined by the "parity" of the numerator and denominator of $z = p/q$, i.e. one of the six equivalence classes under the equivalence relation*

$$\frac{p}{q} \sim \frac{r}{s} \iff ps \equiv qr \pmod 2.$$

Proof. The extended Bianchi[1] group $B[-1] = PGL_2(\mathbb{Z}[i]) \rtimes \langle \mathfrak{c} \rangle$ is a maximal discrete subgroup of $\mathrm{Isom}(\mathcal{H}^3) \cong PSL_2(\mathbb{C}) \rtimes \langle \mathfrak{c} \rangle$, where $\mathfrak{c}$ is complex conjugation. The group Γ is the kernel of the

[1] after Luigi Bianchi, [Bia92]

surjective map

$$B[-1] = PGL_2(\mathbb{Z}[i]) \rtimes \langle \mathfrak{c} \rangle \to PGL_2(\mathbb{Z}[i]/(2))$$

since Γ is in the kernel and both are of index 48 in $B[-1]$ (we have $[B[-1] : \Gamma] = 48$ comparing fundamental domains and $|PGL_2(\mathbb{Z}[i]/(2))| = 48$ by direct computation). Hence Γ preserves parity, e.g.

$$\mathfrak{s}_1\left(\frac{p}{q}\right) = \frac{(1+2i)\bar{p} - 2\bar{q}}{2\bar{p} + \bar{q}(-1+2i)} \equiv \frac{p}{q} \bmod 2.$$

Termination in finite time follows from the following version(s) of the Euclidean algorithm in $\mathbb{Z}[i]$. "Homogenizing" T_A and T_B gives dynamical systems on pairs $(0,0) \neq (p,q) \in \mathbb{Z}[i] \times \mathbb{Z}[i]$ that terminate when $p/q \in \{0, 1, \infty, i, 1+i, \frac{1}{1-i}\}$, i.e. act on pairs via complex conjugation and the matrices implied in the definitions of the $\mathfrak{s}_i$, $\mathfrak{s}_i^\perp$. For instance,

$$T_B(p,q) := (\bar{p}, 2\bar{p} - \bar{q}) \text{ for } \frac{p}{q} \in B_2', \text{ where } T_B(p/q) = \mathfrak{s}_2(p/q) = \frac{\overline{p/q}}{2(\overline{p/q}) - 1}.$$

We'll consider the case of T_B, noting that the proof for T_A is nearly identical:

$$(p,q) \mapsto \begin{cases} s_1(p,q) = ((1+2i)\bar{p} - 2\bar{q}, 2\bar{p} + (2i-1)\bar{q}) & p/q \in B_1' \setminus \{i, 1+i, \frac{1}{1-i}\} \\ s_2(p,q) = (\bar{p}, 2\bar{p} - \bar{q}) & p/q \in B_2' \setminus \{0, 1, \frac{1}{1-i}\} \\ s_3(p,q) = (2\bar{q} - \bar{p}, \bar{q}) & p/q \in B_3' \setminus \{1, 1+i, \infty\} \\ s_4(p,q) = (-\bar{p}, \bar{q}) & p/q \in B_4' \setminus \{0, i, \infty\} \\ s_1^\perp(p,q) = (\bar{p}, \bar{q}) & p/q \in B_1 \setminus \{0, 1, \infty\} \\ s_2^\perp(p,q) = (\bar{p} + 2i\bar{q}, \bar{q}) & p/q \in B_2 \setminus \{i, i+1, \infty\} \\ s_3^\perp(p,q) = (\bar{p}, \bar{q} - 2i\bar{p}) & p/q \in B_3 \setminus \{0, i, \frac{1}{1-i}\} \\ s_4^\perp(p,q) = ((1-2i)\bar{p} + 2i\bar{q}, (2i+1)\bar{q} - 2i\bar{p}) & p/q \in B_4 \setminus \{1, 1+i, \frac{1}{1-i}\}. \end{cases}$$

The inequalities defining the regions B_i, B_i' show that $|q|$ is decreased whenever s_1, s_2, $s_3^\perp$, or $s_4^\perp$ is applied. For example, when applying s_1, the fact that p/q is in the triangle $B_1' \setminus \{i, 1+i, \frac{1}{1-i}\}$ (or the circle A_1 which the inequalities define) shows that $|q|$ is decreased when applying s_1, as follows:

$$\frac{p}{q} \in B_1' \setminus \left\{i, 1+i, \frac{1}{1-i}\right\} \Longrightarrow \left|\frac{p}{q} - \frac{1+2i}{2}\right|^2 < \frac{1}{4} \quad \Longrightarrow \quad |2\bar{p} + (1-2i)\bar{q}| < |q|.$$

Similarly, application of s_3 and $s_2^\perp$ both decrease $|p|$. Applying s_4 maps B'_4 onto $B'_1 \cup B'_2 \cup B_4 \cup B'_3$ (from which one of $|p|$, $|q|$ will be decreased as just discussed). Finally, $s_1^\perp$ maps B_1 onto one of the other seven regions. Hence the algorithm terminates. $\square$

4.4.4 Geometry of the first approximation constant for $\mathbb{Q}(i)$

The purpose of this section and the next is to relate the Super-Apollonian continued fraction algorithm to classical statements of Diophantine approximation. In this section, we give the first value of the Lagrange spectrum for complex approximation by Gaussian rationals. With this as a point for comparison, in the second section we describe the quality of the approximations obtained by the algorithm.

The "good" rational approximations to an irrational $z \in \mathbb{C}$

$$|z - p/q| \leq C/|q|^2$$

are determined by the collection of horoballs

$$B_C(p/q) = \{(z,t) \in H^3 : |z - p/q|^2 + (t - C/|q|^2)^2 \leq C^2/|q|^4\},$$
$$B_C(\infty) = \{(z,t) \in H^3 : t \geq 1/2C\},$$

through which the geodesic $\overrightarrow{\infty z}$ passes (or through which any geodesic $\overrightarrow{wz}$ eventually passes). Over $\mathbb{Q}(i)$, the smallest value of C with the property that every irrational z has infinitely many rational approximations satisfying the above inequality was determined by Ford in [For25]. Here we give a short proof of this fact using the geometry of the ideal octahedron.

Proposition 4.4.2. *Every $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ has infinitely many rational approximations $p/q \in \mathbb{Q}(i)$ such that*

$$\left| z - \frac{p}{q} \right| \leq \frac{C}{|q|^2}, \quad C = \frac{1}{\sqrt{3}} \simeq 0.577350269\ldots$$

The constant $1/\sqrt{3}$ is the smallest possible, as witnessed by $z = \frac{1+\sqrt{-3}}{2}$.

Proof. The value of C for which the horoballs with parameter C based at the ideal vertices $\{0, 1, \infty, i, 1+i, 1/(1-i)\}$ cover the boundary of the fundamental octahedron is easily found to be

$1/\sqrt{3}$ (one need only cover the face with vertices $\{0, 1, \infty\}$, see Figure 4.12). Hence as we follow the geodesic $\overrightarrow{\infty z}$ through the tesselation by octahedra, at least one of the six vertices of each octahedron satisfies the above inequality with $C = 1/\sqrt{3}$, one or two as it enters and one or two as it exits (some of which may coincide). This gives the smallest value of C for which the inequality above has infinitely many solutions for all irrational z, noting the the geodesic from $e^{-\pi i/3}$ to $e^{\pi i/3}$ passes orthogonally through the "centers" of the opposite faces of the octahedra through which it passes. □

The sequence of octahedra we consider in our continued fraction algorithm are not necessarily along the geodesic path, but we do capture all rationals with $|z - p/q| < C/|q|^2$ with $C = 1/(1 + 1/\sqrt{2})$ as detailed in the next section.

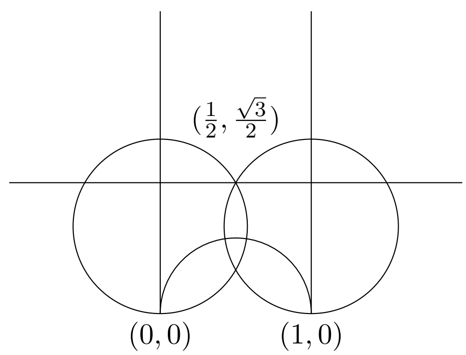

Figure 4.12: Horoball covering of the ideal triangular face with vertices $\{0, 1, \infty\}$ with coordinates (z, t).

4.4.5 Quality of rational approximation

To any complex number we associate six sequences of Gaussian rational approximations by following the inverse orbit of the six vertices of the fundamental octahedron (six points of tangency of our base quadruples R_A, R_B). Namely, if $z = \prod_{i=1}^{\infty} \mathfrak{z}_i = \prod_{i=1}^{\infty} \mathfrak{w}_i = w$ in the two codings, then the convergents $p^A_{n,\alpha}/q^A_{n,\alpha}$, $p^B_{n,\alpha}/q^B_{n,\alpha}$ are given by

$$\frac{p^A_{n,\alpha}}{q^A_{n,\alpha}} = \left(\prod_{i=1}^{n} \mathfrak{w}_i\right)(\alpha),\ \frac{p^B_{n,\alpha}}{q^B_{n,\alpha}} = \left(\prod_{i=1}^{n} \mathfrak{z}_i\right)(\alpha),\ \alpha \in \{0, 1, \infty, i, i+1, 1/(1-i)\},$$

with the property that

$$\lim_{n\to\infty} \frac{p^A_{n,\alpha}}{q^A_{n,\alpha}} = w, \ \lim_{n\to\infty} \frac{p^B_{n,\alpha}}{q^B_{n,\alpha}} = z$$

for all α and $w \in \mathbb{C} \setminus \mathbb{Q}(i)$, $z \in \mathbb{C} \setminus \mathbb{Q}(i)$.

The following theorem is more or less equivalent to a statement about approximation by Schmidt's continued fractions ([Sch75a], Theorem 2.5), where it is stated without proof.

Theorem 4.4.3. *If p/q is such that*

$$|z_0 - p/q| < \frac{C}{|q|^2}, \quad C = \frac{\sqrt{2}}{1+\sqrt{2}} = 0.585786437\ldots,$$

then p/q is a convergent to z_0 (with respect to both T_A and T_B). Moreover, the constant C is the largest possible.

Proof. Note that the Apollonian super-packings associated to the root quadruples R_A, R_B, are invariant under the action of $\mathrm{PSL}_2(\mathbb{Z}[i]) \rtimes \langle \mathfrak{c} \rangle$. Consider the quadruple where p/q first appears as a convergent to z_0, and let $\gamma(z) = \frac{-Qz+P}{-qz+p} \in \mathrm{PSL}_2(\mathbb{Z}[i])$ take this quadruple to the base quadruple (say R_B) with p/q mapping to infinity and infinity mapping to Q/q. For any value of C, the disk of radius $C/|q|^2$ centered at p/q gets mapped by γ to the exterior of the disk of radius $1/C$ centered at Q/q:

$$w = \frac{-Qz+P}{-qz+p} \Rightarrow |z - p/q| = \frac{1}{|w - Q/q||q|^2},$$
$$\frac{C}{|q|^2} \geq |z - p/q| = \frac{1}{|w - Q/q||q|^2} \Rightarrow |w - Q/q| \geq 1/C.$$

Consider the ways in which p/q can first appear as a convergent to z_0 in the sequence of partitions of the plane.

- We might invert into a circle containing z_0. In particular, then, p/q is in the interior of the circle of inversion (since it is its first appearance as a convergent). In this case, by the discussion in 4.4.2, all z inside the circle also include this inversion in their expansion. Therefore, all z inside the circle will have p/q as a convergent. Our goal is to show that the circle of radius $1/|q|^2$ around p/q is contained in the circle of inversion.

Under γ above (perhaps after applying some binary tetrahedral symmetry of the base quadruple), the circles A, B, get mapped to A', B' as in the figures below, with $Q/q = \gamma(\infty)$ lying in the triangle inside B' as shown. The exterior of a disk of radius one centered at Q/q does not meet the interior of B', hence, applying γ^{-1}, the disk of radius $1/|q|^2$ centered at p/q does not meet B. Therefore p/q is a convergent to any z with $|z - p/q| < 1/|q|^2$. See Figure 4.13.

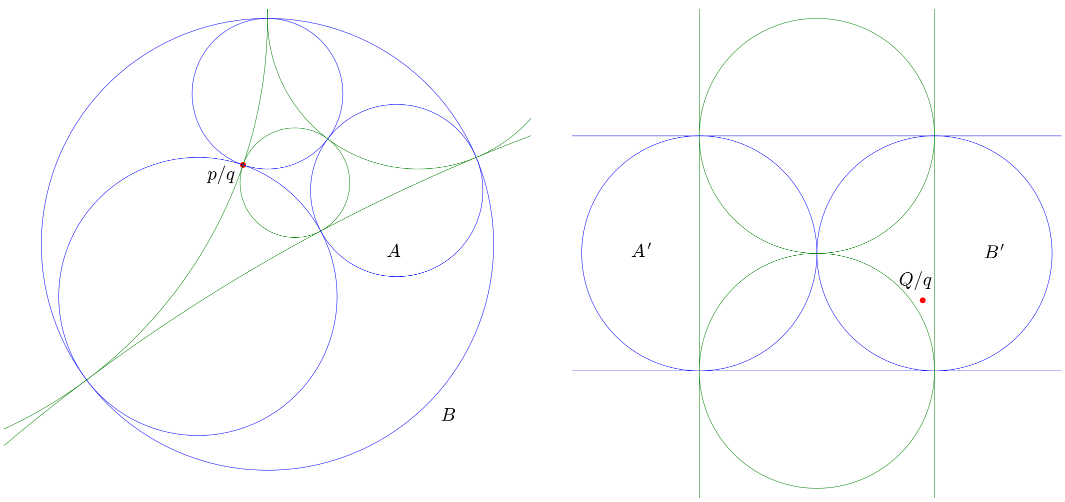

Figure 4.13: Left: The Descartes quadruples (octahedon) when p/q first appears as a convergent when inverting. Right: The result after mapping back to the fundamental octahedron and sending $p/q \to \infty$ and ∞ to Q/q.

- We might swap into a triangle containing z_0 producing p/q as a point of tangency on an edge of this triangle. In the image, z_0 is in the quadrangle with sides formed by A, B, C, D; this is the union of two triangles. The dotted circles in the figures indicate the two possible swaps associated to the initial creation of p/q as a convergent. In this case, any z in the indicated quadrangle will have p/q as a convergent. We aim to show that a circle of radius $C/|q|^2$ around p/q is contained in this region.

  Under γ above (perhaps after applying some binary tetrahedral symmetry of the base quadruple), the circles A, B, C, D, and E are mapped to A', B' C', D' and E', the three

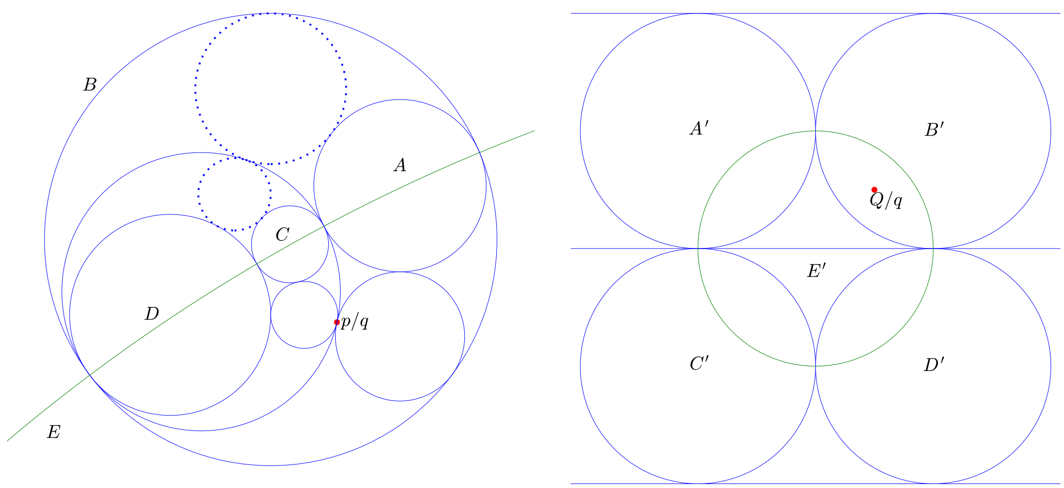

Figure 4.14: Left: The Descartes quadruples (octahedra) when p/q first appears as a convergent when swapping. Right: The result after mapping back to the fundamental octahedron and sending $p/q \to \infty$ and ∞ to Q/q.

circles tangent to p/q are mapped to the lines in the second picture, with $Q/q = \gamma(\infty)$ lying in the intersection of the disks defined by B' and E'. The exterior of any circle of radius $1/C = 1 + 1/\sqrt{2}$ centered inside E' avoids the interiors of A', B', C', and D'. Applying γ^{-1} shows that the disk of radius $C/|q|^2$ around p/q does not meet A, B, C, or D, so that p/q is a convergent to any z with $|z - p/q| < C/|q|^2$. See Figure 4.22.

□

4.4.6 Invertible extension and invariant measures

In this section, we discuss the invertible extension T of T_B, with the property that T^{-1} extends T_A, along with an invariant measure for T. In what follows we adhere to the convention of using w, $\mathfrak{w}$ as coordinates for T_A and z, $\mathfrak{z}$ as coordinates for T_B since we will be using both throughout.

The space of oriented geodesics in $\mathcal{H}^3$, identified with pairs in $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \setminus diag.$ carries an isometry invariant measure

$$|z - w|^{-4} du\, dv\, dx\, dy,\ w = u + iv,\ z = x + iy.$$

We restrict this measure to geodesics in the set

$$\mathcal{G} = \left(\bigcup_i A_i \times B_i\right) \bigcup \left(\bigcup_{i \neq j} A_i' \times B_j\right) \bigcup \left(\bigcup_{i \neq j} A_i \times B_j'\right) \bigcup \left(\bigcup_{i,j} A_i' \times B_j'\right),$$

consisting of geodesics between disjoint A and B regions of the base quadruples (see Figure 4.9). In what follows we use A coordinates for w in the first coordinate and B coordinates for z in the second coordinate. Define $T : \mathcal{G} \to \mathcal{G}$ by

$$T(w, z) = \begin{cases} (\mathfrak{s}_i w, \mathfrak{s}_i z) & z \in B_i,\ \mathfrak{z} = \mathfrak{s}_i \dots \\ (\mathfrak{s}_i^\perp w, \mathfrak{s}_i^\perp z) & z \in B_i',\ \mathfrak{z} = \mathfrak{s}_i^\perp \dots \end{cases}$$

where $\mathfrak{z}$ is the Möbius transformation corresponding to z as described in 4.4.2. Equivalently,

$$T^{-1}(w, z) = \begin{cases} (\mathfrak{s}_i w, \mathfrak{s}_i z) & w \in A_i,\ \mathfrak{w} = \mathfrak{s}_i \dots \\ (\mathfrak{s}_i^\perp w, \mathfrak{s}_i^\perp z) & w \in A_i',\ \mathfrak{w} = \mathfrak{s}_i^\perp \dots \end{cases}.$$

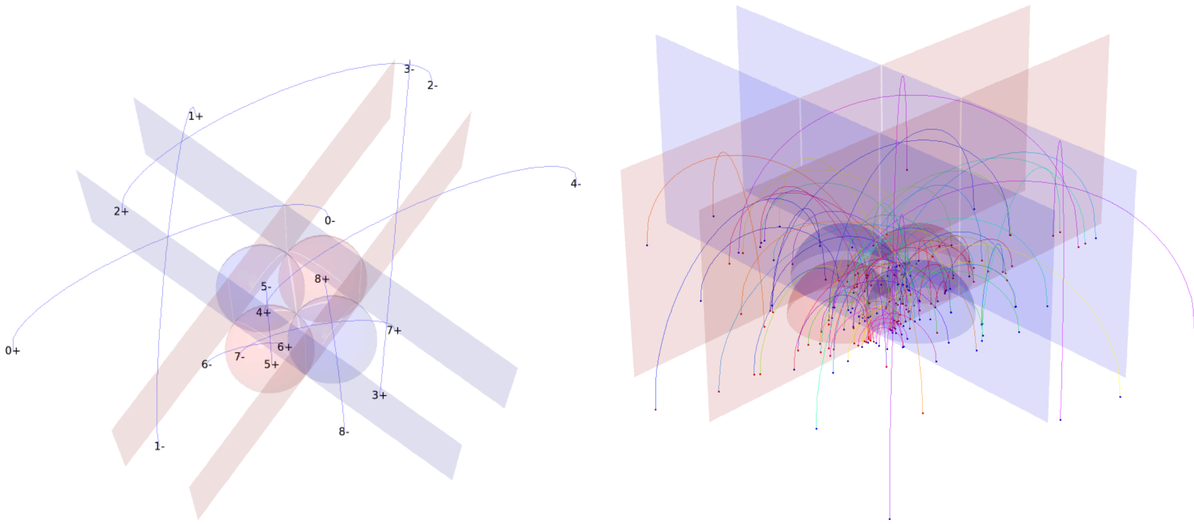

Figure 4.15: At left, eight iterations of the map T, labelled in sequence (with $+$ and $-$ indicating the orientation). At right, 100 iterations of T, with rainbow coloration indicating time. In both pictures, the geodesic planes defining the octahedron are shown.

In other words, T is applying T_B diagonally depending on the second coordinate and T^{-1} is applying T_A diagonally depending on the first coordinate. In terms of the shifts on pairs $(\mathfrak{w}, \mathfrak{z}) = (\prod_i \mathfrak{w}_i, \prod_i \mathfrak{z}_i)$ corresponding to (w, z), we have

$$T(w, z) = (\mathfrak{z}_1 \mathfrak{w}, \prod_{i=2}^{\infty} \mathfrak{z}_i) = (\mathfrak{z}_1 w, T_B(z)), \;\; T^{-1}(w, z) = (\prod_{i=2}^{\infty} \mathfrak{w}_i, \mathfrak{w}_1 \mathfrak{z}) = (T_A(w), \mathfrak{w}_1 z).$$

See Figure 4.15 for a visualization of the invertible extension.

Define the following regions (see Figure 4.16):

$$\mathcal{A}_i = B_i \cup (\cup_{j \neq i} B'_j)$$

$$\mathcal{A}'_i = (\cup_j B'_j) \cup (\cup_{j \neq i} B_j)$$

$$\mathcal{B}_i = A_i \cup (\cup_{j \neq i} A'_j)$$

$$\mathcal{B}'_i = (\cup_j A'_j) \cup (\cup_{j \neq i} A_j)$$

Here $\mathcal{A}_i$ consists of the B regions not intersecting A_i, and so on.

Theorem 4.4.4. *The function $T : \mathcal{G} \to \mathcal{G}$ is a measure-preserving bijection. Consequently, the push-forward of this measure onto the first or second coordinate gives invariant measures μ_A and*

μ_B for T_A and T_B. Specifically, we have

$$d\mu_A(w) = f_A(w)\ du\ dv = \begin{cases} f_{A_i}(u,v)\ du\ dv = du\ dv \int_{\mathcal{A}_i} |z-w|^{-4}\ dx\ dy, & w \in A_i \\ f_{A_i'}(u,v)\ du\ dv = du\ dv \int_{\mathcal{A}_i'} |z-w|^{-4}\ dx\ dy, & w \in A_i' \end{cases}$$

and

$$d\mu_B(z) = f_B(z)\ dx\ dy = \begin{cases} f_{B_i}(x,y)\ dx\ dy = dx\ dy \int_{\mathcal{B}_i} |z-w|^{-4}\ du\ dv, & z \in B_i \\ f_{B_i'}(x,y)\ dx\ dy = dx\ dy \int_{\mathcal{B}_i'} |z-w|^{-4}\ du\ dv, & z \in B_i' \end{cases}.$$

See Figure 4.17 for a graph of the invariant density f_B.

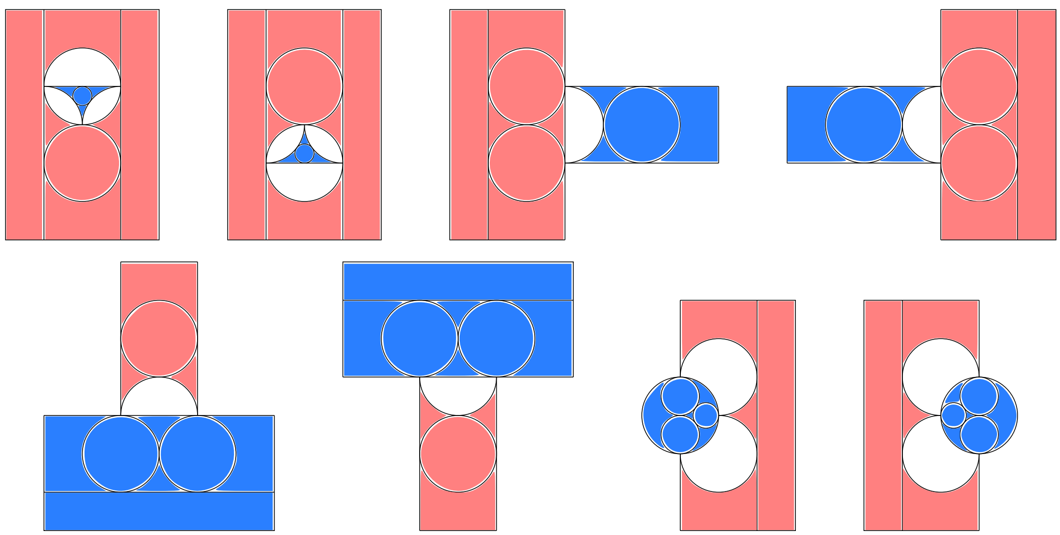

Figure 4.16: Top, left to right: regions $\mathcal{B}_i' \times B_i'$, $i = 1,2,3,4$. Bottom, left to right: regions $\mathcal{B}_i \times B_i$, $i = 1,2,3,4$. Shown red×blue with subdivisions.

Proof. Note that $\mathcal{G} = \cup_i(A_i \times \mathcal{A}_i \cup A_i' \times \mathcal{A}_i') = \cup_i(\mathcal{B}_i \times B_i \cup \mathcal{B}_i' \times B_i')$, which can be seen by in Figure 4.16. Since

$$T(\mathcal{B}_i \times B_i) = A_i' \times \mathcal{A}_i',$$

$$T(\mathcal{B}_i' \times B_i') = A_i \times \mathcal{A}_i,$$

we immediately get that T is a bijection. As T is a bijection defined piecewise by isometries, it preserves the measure described above. $\square$

Computing the relevant integrals in Theorem 4.4.4 gives $\pi/4$ times hyperbolic area on the triangular regions B_i':

$$f_B(x,y) = \begin{cases} \frac{\pi}{4\left(\frac{1}{4}-d^2\right)^2} & z \in B_1',\ d^2 = \left(x-\frac{1}{2}\right)^2 + (y-1)^2 \\ \frac{\pi}{4\left(\frac{1}{4}-d^2\right)^2} & z \in B_2',\ d^2 = \left(x-\frac{1}{2}\right)^2 + y^2 \\ \frac{\pi}{4(1-x)^2} & z \in B_3' \\ \frac{\pi}{4x^2} & z \in B_4' \end{cases},$$

and on the circular regions B_i we have

$$f_B(x,y) = \begin{cases} H(x,y) & z \in B_1 \\ H(x,1-y) & z \in B_2 \\ G(x,y) & z \in B_3 \\ G(1-x,y) & z \in B_4 \end{cases},$$

where

$$\begin{aligned} H(x,y) &= h(x,y) + h(1-x,y) + h(x^2-x+y^2,y), \\ G(x,y) &= h(x,y^2-y+x^2) + h(x^2-x+y^2,y^2-y+x^2) + h(x^2-x+(1-y)^2,y^2-y+x^2), \\ h(x,y) &= \frac{\arctan(x/y)}{4x^2} - \frac{1}{4xy}. \end{aligned}$$

Furthermore, we have the relationship

$$f_A(w) = f_B(\rho w) = f_B(\mathfrak{d} w)$$

where ρ is rotation by $\pi/2$ around $1/(1-i)$ and $\mathfrak{d}$ is the isometry switching opposite faces of the octahedron

$$\mathfrak{d} = \frac{\bar{z}-1+i}{(1-i)\bar{z}+i} = \mathfrak{d}^{-1},\ \mathfrak{s}_i^{\perp} = \mathfrak{d}\mathfrak{s}_i\mathfrak{d},$$

(the Möbius equivalent of the "duality" operator from [GLM$^+$06a]). The measures μ_A, μ_B have S_3 symmetry on each of the A_i, A_i', B_i, B_i'. For instance the isometries permuting $\{0,1,\infty\}$ on A_1', B_1, preserve the measure (generators shown for a transposition and three-cycle)

$$(0,1) \sim -\bar{z}+1,\ (0,1,\infty) \sim \frac{-1}{z-1}.$$

The measures also have S_4 symmetry on $\mathbb{C}$, permuting the pairs $\{A_i, A'_i\}$, $\{B_i, B'_i\}$ (transpositions $(i, i+1)$ shown)

$$(1,2) \sim \bar{z} + i,\ (2,3) \sim \frac{1}{\bar{z}},\ (3,4) \sim -\bar{z} + 1.$$

The total measure assigned to each of the A_i, A'_i, B_i, B'_i, is $\pi^2/4$, so to normalize μ_A, μ_B, we divide by $2\pi^2$. All of the above should be compared with Nakada's extension of Schmidt's system, described in chapter 3.4.

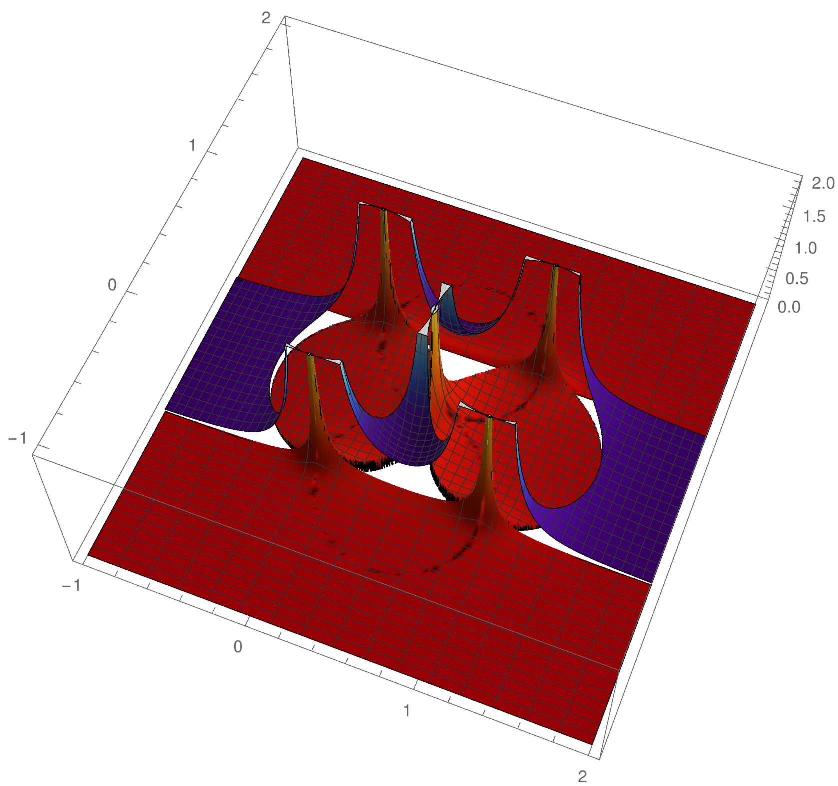

Figure 4.17: Graph of $f_B(z)$, the invariant density for T_B.

The following lemmas compare the measures of some Farey circles and triangles via the involutions $\perp$ and $^{-1}$, simplifying computations. For use in the proofs, we note the equality of regions

$$\mathfrak{s}_i^{\perp}\mathcal{B}_i = A_i' = \mathfrak{d}B_i',$$
$$\mathfrak{s}_i\mathcal{B}_i' = A_i = \mathfrak{d}B_i, \tag{4.1}$$
$$\mathfrak{s}_i\mathcal{A}_i = B_i' = \mathfrak{d}A_i',$$
$$\mathfrak{s}_i^{\perp}\mathcal{A}_i' = B_i = \mathfrak{d}A_i.$$

Lemma 4.4.5. *For $\mathfrak{m}$, $\mathfrak{n}$, in normal form (associated to T_B and T_A respectively), there are equalities of measure*

$$\mu_B(F_B(\mathfrak{m})) = \mu_B(F_B(\mathfrak{m}^{\perp})), \quad \mu_A(F_A(\mathfrak{n})) = \mu_A(F_A(\mathfrak{n}^{\perp})).$$

Proof. Consider the case where $\mathfrak{m} = \mathfrak{s}_i \dots \mathfrak{s}_j$ so that $F_B(\mathfrak{m}) = \mathfrak{m}\mathfrak{s}_j B_j' \subseteq B_i'$ and $F_B(\mathfrak{m}^{\perp}) = \mathfrak{m}^{\perp}\mathfrak{s}_i^{\perp} B_i \subseteq B_j$ (all other cases are analogous) and let $\omega = |z-w|^{-4}\, du\, dv\, dx\, dy$ be the invariant form on the space of geodesics. Note that $\mathfrak{m}^{\perp} = \mathfrak{d}\mathfrak{m}^{-1}\mathfrak{d}$. We have

$$\mu_B(F_B(\mathfrak{m})) = \int_{\mathcal{B}_i'} \int_{F_B(\mathfrak{m})} \omega = \int_{\mathcal{B}_i'} \int_{\mathfrak{m}\mathfrak{s}_j B_j'} \omega$$

while

$$\begin{aligned}\mu_B(F_B(\mathfrak{m}^{\perp})) &= \int_{\mathcal{B}_j} \int_{F(\mathfrak{m}^{\perp})} \omega = \int_{\mathcal{B}_j} \int_{\mathfrak{m}^{\perp}\mathfrak{s}_i^{\perp} B_i} \omega = \int_{\mathcal{B}_j} \int_{\mathfrak{d}\mathfrak{m}^{-1}\mathfrak{d}\mathfrak{s}_i^{\perp} B_i} \omega = \int_{\mathfrak{m}\mathfrak{d}\mathcal{B}_j} \int_{\mathfrak{d}\mathfrak{s}_i^{\perp} B_i} \omega \\ &= \int_{\mathfrak{m}\mathfrak{d}\mathcal{B}_j} \int_{\mathfrak{s}_i\mathfrak{d}B_i} \omega = \int_{\mathfrak{m}\mathfrak{s}_j B_j'} \int_{\mathfrak{s}_i A_i} \omega = \int_{\mathfrak{m}\mathfrak{s}_j B_j'} \int_{\mathcal{B}_i'} \omega.\end{aligned}$$

Here we are using the fact that $\omega = |z-w|^{-4}\, du\, dv\, dx\, dy$ is the invariant form and the relations in (4.1). $\square$

Lemma 4.4.6. *For $\mathfrak{m}$, $\mathfrak{n}$ in normal form (associated to T_B and T_A respectively), there are equalities of measure*

$$\mu_B(F_B(\mathfrak{m})) = \mu_A(F_A(\mathfrak{m}^{-1})), \quad \mu_A(F_A(\mathfrak{n})) = \mu_B(F_B(\mathfrak{n}^{-1})).$$

Proof. Consider the case where $\mathfrak{m} = \mathfrak{s}_i \dots \mathfrak{s}_j$ so that $F_B(\mathfrak{m}) = \mathfrak{m}\mathfrak{s}_j B_j' \subseteq B_i'$ and $F_A(\mathfrak{m}^{-1}) = \mathfrak{m}^{-1}\mathfrak{s}_i A_i \subseteq A_j$ (all other cases are analogous) and let $\omega = |z-w|^{-4}\, du\, dv\, dx\, dy$ be the invariant form on the space of geodesics. Then

$$\mu_B(F(\mathfrak{m})) = \int_{\mathcal{B}'_i}\int_{F_B(m)} \omega = \int_{\mathcal{B}'_i}\int_{\mathfrak{m}\mathfrak{s}_j B'_j} \omega = \int_{\mathfrak{s}_i A_i}\int_{\mathfrak{m}\mathcal{A}_j} \omega = \int_{\mathfrak{m}^{-1}\mathfrak{s}_i A_i}\int_{\mathcal{A}_j} \omega$$

while

$$\mu_A(F(\mathfrak{m}^{-1})) = \int_{F_A(\mathfrak{m}^{-1})}\int_{\mathcal{A}_j} \omega = \int_{\mathfrak{m}^{-1}\mathfrak{s}_i A_i}\int_{\mathcal{A}_j} \omega.$$

Here we are using the fact that $\omega = |z-w|^{-4}\, du\, dv\, dx\, dy$ is the invariant form and the relations in (4.1). $\square$

4.5 Cubeoctahedral continued fractions over $\mathbb{Q}(\sqrt{-2})$

In this section, we repeat some of the arguments of the last section for the ideal right-angled regular cubeoctahedron with vertices

$$0,\ 1,\ \infty,\ \frac{-1}{\sqrt{-2}},\ \frac{1+\sqrt{-2}}{2},\ \frac{\sqrt{-2}-1}{\sqrt{-2}},\ \sqrt{-2},\ 1+\sqrt{-2},$$
$$\frac{1}{1-\sqrt{-2}},\ \frac{\sqrt{-2}}{1+\sqrt{-2}},\ \frac{-1+\sqrt{-2}}{1+\sqrt{-2}},\ \frac{2}{1-\sqrt{-2}}.$$

Hyperbolic 3-space can be tessellated by this polyhedron, and the tessellation is invariant under the group $GL_2(\mathbb{Z}[\sqrt{-2}])$. The associated continued fractions have applications to Diophantine approximation over the field $\mathbb{Q}(\sqrt{-2})$.

4.5.1 Dual pair of dynamical systems, invertible extension, and invariant measures

At the end of the introduction to [Sch11], Schmidt remarks that there is an ergodic theory for his continued fractions over $\mathbb{Q}(\sqrt{-2})$ similar to that over $\mathbb{Q}(i)$ in [Sch82]. Here is a version of that in terms of inversions in eight circles (with cubic tangency) and six 'dual' circles (with octahedral tangency).

Consider the following 14 Möbius transformations defined over $\mathbb{Z}[\sqrt{-2}]$

$$\mathfrak{s}_1 = \frac{(1+2\sqrt{-2})\bar{z}-4}{2\bar{z}-1+2\sqrt{-2}},\ \mathfrak{s}_2 = \frac{\bar{z}}{2\bar{z}-1},\ \mathfrak{s}_3 = -\bar{z}+2,$$

$$\mathfrak{s}_4 = -\bar{z},\ \mathfrak{s}_5 = \frac{(1+2\sqrt{-2})\bar{z}-2}{4\bar{z}-1+2\sqrt{-2}},\ \mathfrak{s}_6 = \frac{(3+2\sqrt{-2})\bar{z}-4}{4\bar{z}-3+2\sqrt{-2}},$$

$$\mathfrak{t}_1 = \bar{z}+2\sqrt{-2},\ \mathfrak{t}_2 = \bar{z},\ \mathfrak{t}_3 = \frac{(5-2\sqrt{-2})\bar{z}+4\sqrt{-2}}{-4\sqrt{-2}\bar{z}+5+2\sqrt{-2}},\ \mathfrak{t}_4 = \frac{(3-2\sqrt{-2})\bar{z}+2\sqrt{-2}}{-4\sqrt{-2}\bar{z}+3+2\sqrt{-2}},$$

$$\mathfrak{t}_5 = \frac{(3-2\sqrt{-2})\bar{z}+4\sqrt{-2}}{-2\sqrt{-2}\bar{z}+3+2\sqrt{-2}},\ \mathfrak{t}_6 = \frac{\bar{z}}{-2\sqrt{-2}\bar{z}+1},\ \mathfrak{t}_7 = \frac{(1-2\sqrt{-2})\bar{z}+2\sqrt{-2}}{-2\sqrt{-2}\bar{z}+1+2\sqrt{-2}},\ \mathfrak{t}_8 = \frac{3\bar{z}+2\sqrt{-2}}{-2\sqrt{-2}\bar{z}+3},$$

inversions in the circles whose interiors are A_i and B_i (see Figures 4.18, 4.19). These generate a discrete group Γ of isometries of hyperbolic three-space (of finite covolume), reflections in the sides of an ideal, right-angled cubeoctahedron. The cubeoctahedral reflection group Γ is the kernel of the map

$$PGL_2(\mathbb{Z}[\sqrt{-2}]) \rtimes \langle \mathfrak{c} \rangle \to PGL_2(\mathbb{Z}[\sqrt{-2}]/(2)),$$

($\mathfrak{c}$ complex conjugation), similar to the Gaussian situation of the previous section.

We define two dynamical systems, $T_A, T_B : \mathbb{C} \to \mathbb{C}$

$$T_A(w) = \begin{cases} \mathfrak{s}_i w & w \in A_i \\ \mathfrak{t}_i w & w \in A_i' \end{cases}$$

$$T_B(z) = \begin{cases} \mathfrak{t}_i z & z \in B_i \\ \mathfrak{s}_i z & z \in B_i' \end{cases}$$

and use these to coordinatize the plane in two different ways using the alphabet $\{\mathfrak{s}_j, \mathfrak{t}_j\}$. We write $w = \mathfrak{w} = \prod_{i=1}^{\infty} \mathfrak{w}_i$ if $T_A^i(w) = \mathfrak{w}_i T_A^{i-1} w$ and similarly $z = \mathfrak{z} = \prod_{i=1}^{\infty} \mathfrak{z}_i$ if $T_B^i(z) = \mathfrak{z}_i T_B^{i-1} z$. With these coordinates, T_A, T_B are the shift maps. [The words are in "normal form" as described in 4.2.3] Both T_A and T_B have 12 fixed points (the vertices of the cubeoctahedron) and rational points go to them in finite time, depending once again on their character modulo 2 (12 coprime pairs from $\mathbb{Z}[\sqrt{-2}]/(2)$). See Figure 4.20 for the A and B "super-packings."

We now define an invertible extension T of T_A and T_B (T will extend T_B and T^{-1} will extend T_A), defined on a subset of the space of geodesics of hyperbolic 3-space $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \setminus \mathrm{diag}$. Define

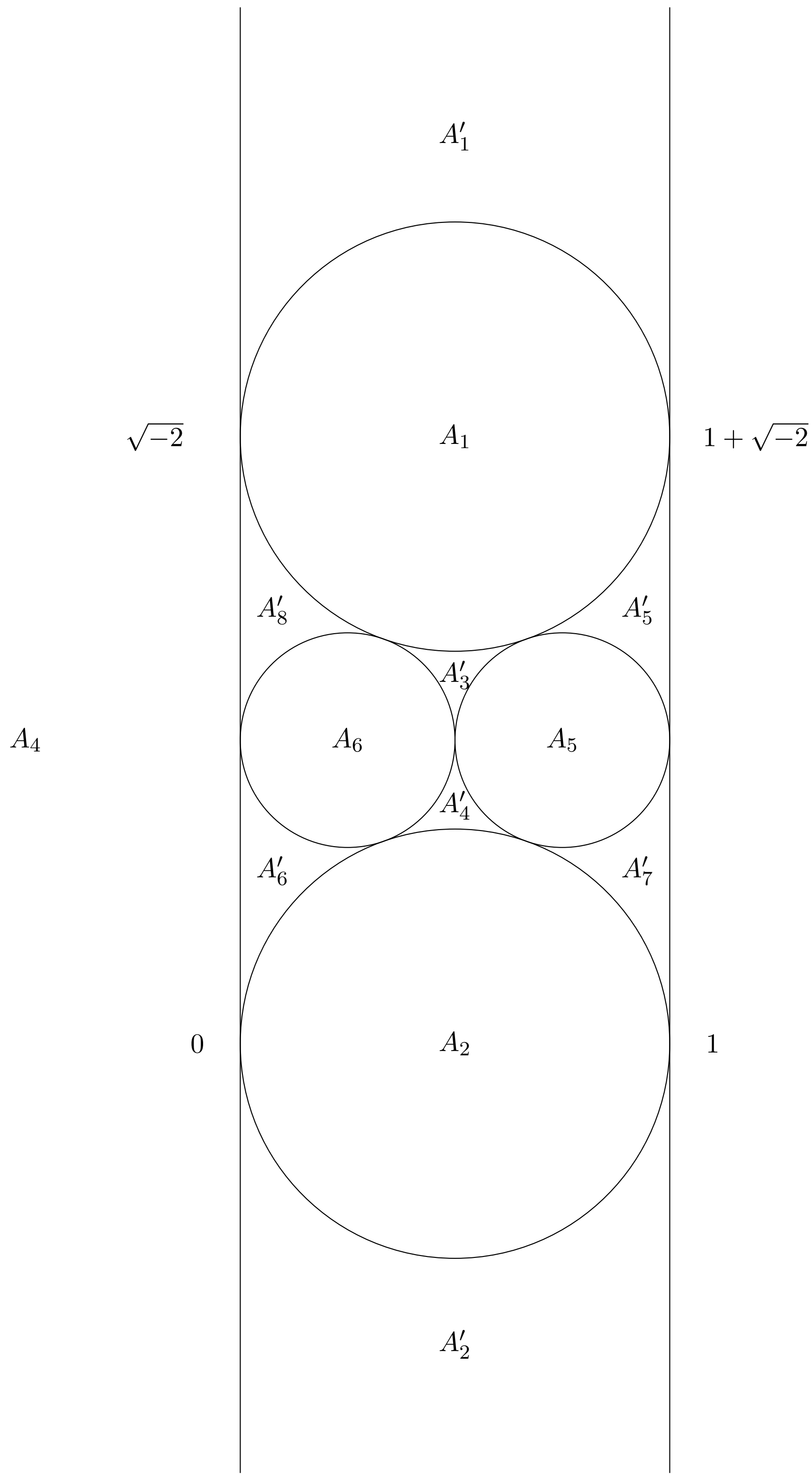

Figure 4.18: Regions in the definition of T_A, boundaries given by the fixed circles of the $\mathfrak{s}_i$.

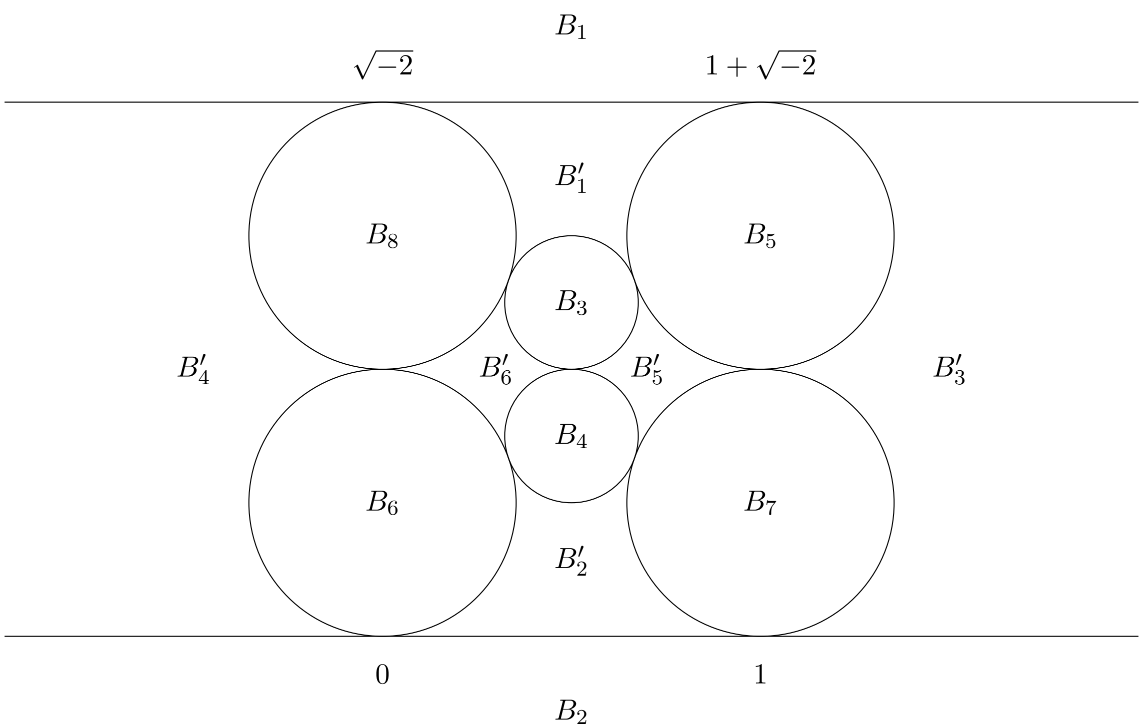

Figure 4.19: Regions in the definition of T_B, boundaries given by the fixed circles of the $\mathfrak{t}_i$.

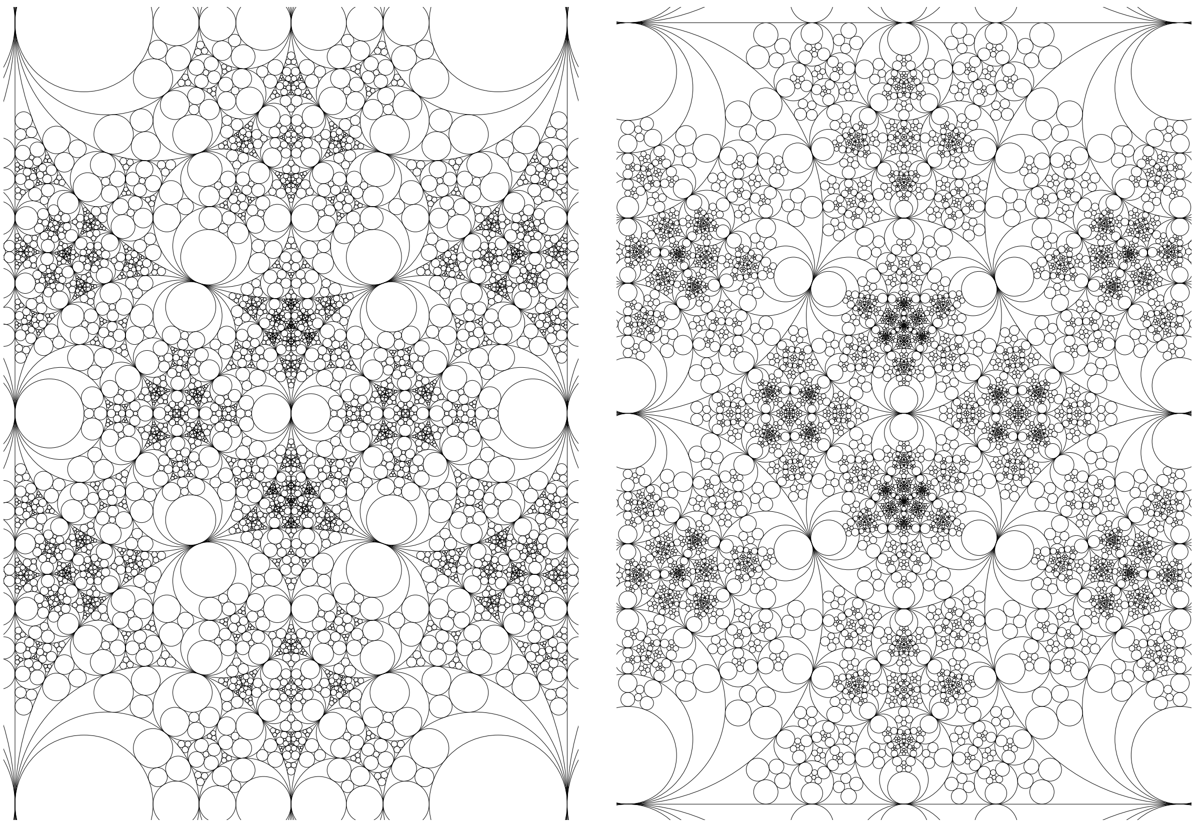

Figure 4.20: Dual super-packings for the cubeoctahedron associated to $\mathbb{Q}(\sqrt{-2})$.

the following groupings of the regions A_i, A_i', B_i, B_i'

$$\mathcal{A}_i = (\cup_{B_j' \cap A_i = \emptyset} B_j') \cup (\cup_{B_j \cap A_i = \emptyset} B_j)$$

$$\mathcal{A}_i' = (\cup_{B_j' \cap A_i' = \emptyset} B_j') \cup (\cup_{B_j \cap A_i' = \emptyset} B_j)$$

$$\mathcal{B}_i = (\cup_{A_j' \cap B_i = \emptyset} A_j') \cup (\cup_{A_j \cap B_i = \emptyset} A_j)$$

$$\mathcal{B}_i' = (\cup_{A_j' \cap B_i' = \emptyset} A_j') \cup (\cup_{A_j \cap B_i' = \emptyset} A_j)$$

(i.e. for each A region $\mathcal{A}$ is the union of all the B regions disjoint from it, similarly for the $\mathcal{B}$). The space of geodesics on which T will be defined is

$$\mathcal{G} = (\cup_i \mathcal{A}_i \times A_i) \cup (\cup_i \mathcal{A}_i' \times A_i') = (\cup_i \mathcal{B}_i \times B_i) \cup (\cup_i \mathcal{B}_i' \times B_i')$$

with

$$T(w,z) = (\mathfrak{z}_1 \prod_{j=1}^{\infty} \mathfrak{w}_j, \prod_{j=2}^{\infty} \mathfrak{z}_j) = \begin{cases} (\mathfrak{t}_i w, \mathfrak{t}_i z) = (\mathfrak{t}_i w, T_B(z)) & z \in B_i \\ (\mathfrak{s}_i w, \mathfrak{s}_i z) = (\mathfrak{s}_i w, T_B(z)) & z \in B_i' \end{cases}$$

or equivalently

$$T^{-1}(w,z) = (\prod_{j=2}^{\infty} \mathfrak{w}_j, \mathfrak{w}_1 \prod_{j=1}^{\infty} \mathfrak{z}_j) = \begin{cases} (\mathfrak{s}_i w, \mathfrak{s}_i z) = (T_A(w), \mathfrak{s}_i z) & w \in A_i \\ (\mathfrak{t}_i w, \mathfrak{t}_i z) = (T_A(w), \mathfrak{t}_i z) & w \in A_i' \end{cases}.$$

One can verify that T is a bijection

$$T(\mathcal{B}_i \times B_i) = A_i' \times \mathcal{A}_i',$$

$$T(\mathcal{B}_i' \times B_i') = A_i \times \mathcal{A}_i$$

(noting $\mathcal{G} = \cup_i (A_i \times \mathcal{A}_i \cup A_i' \times \mathcal{A}_i') = \cup_i (\mathcal{B}_i \times B_i \cup \mathcal{B}_i' \times B_i')$). The space of geodesics has an isometry invariant measure $|z-w|^{-4} dudvdxdy$ ($w = u + iv$, $z = x + iy$) and since T is a bijection defined piecewise by isometries, this measure is T-invariant on $\mathcal{G}$. The pushforward of this measure onto the first and second coordinates will provide invariant measures μ_A, μ_B for T_A and T_B

$$d\mu_A = f_A(w) dudv = \begin{cases} dudv \int_{\mathcal{A}_i} |z-w|^{-4} dxdy & w \in A_i \\ dudv \int_{\mathcal{A}_i'} |z-w|^{-4} dxdy & w \in A_i' \end{cases},$$

$$d\mu_B = f_B(z)dxdy = \begin{cases} dxdy \int_{\mathcal{B}_i} |z-w|^{-4} dudv & w \in B_i \\ dxdy \int_{\mathcal{B}'_i} |z-w|^{-4} dudv & w \in B'_i \end{cases}.$$

Computing these integrals on the triangular (A'_i) and quadrangular (B'_i) regions gives a multiple of hyperbolic area

$$f_A(w) = \begin{cases} \frac{\pi}{4(v-\sqrt{2})^2} & w \in A'_1 \\ \frac{\pi}{4v^2} & w \in A'_2 \end{cases},$$

$$f_A(w) = \frac{\pi\rho^2}{(\rho^2-d^2)^2},\ d,\rho = \begin{cases} d = |w-(1/2+5\sqrt{-2}/8)|, \rho = \sqrt{2}/8 & w \in A'_3 \\ d = |w-(1/2+3\sqrt{-2}/8)|, \rho = \sqrt{2}/8 & w \in A'_4 \\ d = |w-(1+3\sqrt{-2}/4)|, \rho = \sqrt{2}/4 & w \in A'_5 \\ d = |w-3\sqrt{-2}/4|, \rho = \sqrt{2}/4 & w \in A'_6 \\ d = |w-(1+\sqrt{-2}/4)|, \rho = \sqrt{2}/4 & w \in A'_7 \\ d = |w-\sqrt{-2}/4|, \rho = \sqrt{2}/4 & w \in A'_8 \end{cases},$$

$$f_B(z) = \begin{cases} \frac{\pi}{4(x-1)^2} & z \in B'_3 \\ \frac{\pi}{4x^2} & z \in B'_4 \end{cases},$$

$$f_B(z) = \frac{\pi\rho^2}{(\rho^2-d^2)^2},\ d,\rho = \begin{cases} d = |z-(1/2+\sqrt{-2})|, \rho = 1/2 & w \in B'_1 \\ d = |z-1/2|, \rho = 1/2 & w \in B'_2 \\ d = |z-(1/2+\sqrt{-2})|, \rho = 1/4 & w \in B'_5 \\ d = |z-(1/2+\sqrt{-2})|, \rho = 1/4 & w \in B'_6 \end{cases},$$

with the mass of each piece being $\mu_A(A'_i) = \pi^2/4$, $\mu_B(B'_i) = \pi^2/2$.

On the circular regions A_i, B_i, we have

$$f_A(w) = \begin{cases} G(u,v) & w \in A_4 \\ G(\mathfrak{m}w)\|\mathfrak{m}w\| & w \in A_i \end{cases},$$

$$f_B(z) = \begin{cases} H(x,y) & z \in B_2 \\ H(\mathfrak{n}z)\|\mathfrak{n}z\| & z \in B_i \end{cases},$$

where

$$h(a,b) = \frac{\arctan(a/b)}{4a^2} - \frac{1}{4ab},$$
$$H(a,b) = h(a,b) + h(1-a,b) + h(a^2-a+b^2,b),$$
$$G(a,b) = H(b/\sqrt{2}, a/\sqrt{2}) + H((b-1)/\sqrt{2}, a/\sqrt{2}),$$

and the $\mathfrak{m}$ and $\mathfrak{n}$ are chosen to take $A_i \times \mathcal{A}_i$ or $\mathcal{B}_i \times B_i$ to $\mathcal{A}_4 \times A_4$ or $\mathcal{B}_2 \times B_2$. The integrals are obtained by noting that for any Möbius transformation $\mathfrak{m}$

$$\phi(z) = \int_{\mathfrak{m}R} |z-w|^{-4} du dv \Rightarrow \phi(\mathfrak{m}z)\|\mathfrak{m}z\| = \int_R |z-w|^{-4} du dv.$$

The following act as permutations of $A_i \times \mathcal{A}_i$ or $\mathcal{B}_i \times B_i$ and generate the symmetries (octahedral/cubic):

$$\mathfrak{m}_{(1326)} = \frac{1}{w-\sqrt{-2}},\ \mathfrak{m}_{(165)} = \frac{-1}{w-1},\ \mathfrak{m}_{(12)} = \bar{w} + \sqrt{-2},\ \mathfrak{m}_{(34)(56)} = -\bar{w} + 1,$$

$$\mathfrak{n}_{(3574)} = \frac{1}{z-\sqrt{-2}},\ \mathfrak{n}_{(167)} = \mathfrak{n}_{(458)} = \frac{-1}{z-1},\ \mathfrak{n}_{(12)(34)(57)(68)} = \bar{z} + \sqrt{-2},\ \mathfrak{n}_{(58)(67)} = -\bar{z} + 1.$$

The mass of the circular regions is $\mu_A(A_i) = \pi^2/2$, $\mu_B(B_i) = \pi^2/4$. Hence the total masses are $\mu_A(\mathbb{C}) = 5\pi^2$, $\mu_B(\mathbb{C}) = 11\pi^2/2$.

The "Apollonian" structure of a group isomorphic to the group generated by the "swaps" $\mathfrak{s}_i$ (and similar groups for the other Euclidean imaginary quadratic fields) is explored in [Sta18].

4.5.2 Geometry of the first approximation constant for $\mathbb{Q}(\sqrt{-2})$

Recall that the "good" rational approximations to an irrational $z \in \mathbb{C}$

$$|z - p/q| \leq C/|q|^2$$

are determined by the collection of horoballs

$$B_C(p/q) = \{(z,t) \in H^3 : |z-p/q|^2 + (t - C/|q|^2)^2 \leq C^2/|q|^4\},$$
$$B_C(\infty) = \{(z,t) \in H^3 : t \geq 1/2C\},$$

through which the geodesic $\overrightarrow{\infty z}$ passes (or through which any geodesic $\overrightarrow{wz}$ eventually passes). Over $\mathbb{Q}(\sqrt{-2})$, the smallest value of C with the property that every irrational z has infinitely many rational approximations satisfying the above inequality was determined by Perron in [Per33]. Here we give a short proof of this fact using the geometry of the ideal cubeoctahedron.

Proposition 4.5.1. *Every* $z \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-2})$ *has infinitely many rational approximations* $p/q \in \mathbb{Q}(\sqrt{-2})$ *such that*

$$\left| z - \frac{p}{q} \right| \leq \frac{C}{|q|^2}, \quad C = \frac{1}{\sqrt{2}}.$$

The constant $1/\sqrt{2}$ *is the smallest possible, as witnessed by* $z = \frac{1+i}{\sqrt{2}}$.

Proof. $GL_2(\mathbb{Z}[\sqrt{-2}])$ acts transitively on the faces of the same shape, so we only need cover the triangular face with vertices $\{0, 1, \infty\}$ and the square face that we identify with the ideal quadrilateral with vertices $\{0, -1/\sqrt{-2}, \sqrt{-2}, \infty\}$. The value of C needed for the former is $1/\sqrt{3}$ and for the latter is $1/\sqrt{2}$ (see Figure 4.21).

The point not covered by the open horoballs of parameter $1/\sqrt{2}$ on the square face above the imaginary axis is $(-1/\sqrt{-2}, 1/\sqrt{2})$. An exceptional geodesic entering orthogonally at this point has radius $1/\sqrt{2}$ and exits orthognally through the opposite square face at the point $(\frac{2}{3} + \frac{i}{\sqrt{2}}, \frac{1}{3\sqrt{2}})$. This geodesic has irrational endpoints $\frac{\pm 1+i}{\sqrt{2}}$. Once again, shrinking the horoballs allows this geodesic through, so that $C = 1/\sqrt{2}$ is optimal. $\square$

4.5.3 Quality of approximation

We now consider the quality of approximation of the algorithms above. The 12 sequences of convergents (orbits of the 12 vertices of the cubeoctahedron) to a number z_0 include all good rational approximations as described in the next theorem. The proof follows the same reasoning used in 4.4.3, but the constant disagrees with that given (without proof) in [Sch11]. There ([Sch11] Theorem 2.4) the constant is $C = \frac{2\sqrt{2}}{1+\sqrt{17}}$, smaller than the constant we give below. While our algorithms and those of [Sch11] are both based on the same circle packings, we keep track of

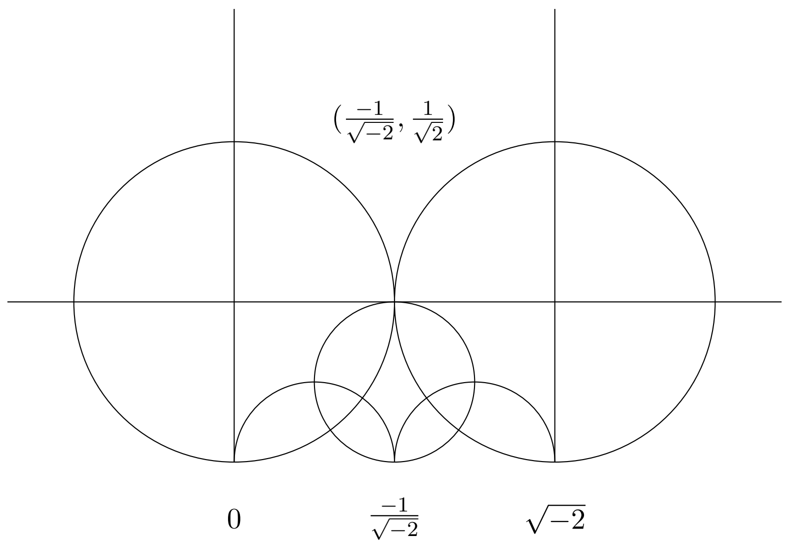

Figure 4.21: Horoball covering of the ideal square face with vertices $\{0, -1/\sqrt{-2}, \sqrt{-2}, \infty\}$ with coordinates (z, t).

more approximations (12 sequences of convergents) than he does (the 5 sequences of convergents necessary to describe the Farey sets of the partition), which explains the discrepancy.

Theorem 4.5.2. *If p/q is such that*

$$|z_0 - p/q| < \frac{C}{|q|^2}, \quad C = \frac{2\sqrt{2}}{1+\sqrt{2}+\sqrt{3}} = 0.682162754\ldots,$$

then p/q is a convergent to z_0 (with respect to both T_A and T_B). Moreover, the constant C is the largest possible.

Proof. The proof follows the reasoning of 4.4.3. We consider when a rational p/q appears for the first time when approximating $z_0 \in \mathbb{C}$, send p/q to infinity and the boundary circles for the cubeoctahedron to the 'basic' configuration, and see what happens. For brevity, we focus on the worst case scenario when p/q arises from 'swapping' into a polygonal region in two different ways, as in Figure 4.22. By sending p/q to ∞ by an appropriate $\gamma \in B[-2] = GL_2(\mathbb{Z}[\sqrt{-2}]) \rtimes \langle \mathfrak{c} \rangle$ (i.e. send our approximating cubeoctahedron back to the fundamental cubeoctahedron via the inverse of T_A^n or T_B^n if p/q is an nth convergent to z_0, then permuting the fundamental cubeoctahedron) we can send the arrangments of Figure 4.22 to the base configurations of Figure 4.23. A neighborhood of p/q of the form

$$\left| z - \frac{p}{q} \right| \leq \frac{C}{|q|^2}$$

(say containing z_0) is now a neighborhood of infinity of the form

$$|z - \gamma(\infty)| \geq \frac{1}{C}.$$

The point $\gamma(\infty)$ is constrained to lie in the shaded region of Figure 4.23, say by considering the black circle and the circles A and R respectively. Since we swapped into the polygonal regions, z_0 must not be in the disks

$$A,\ B,\ C,\ D \text{ (for } T_A),\ R,\ S,\ T,\ U,\ V,\ W \text{ (for } T_B),$$

of Figure 4.22 (oriented with the labels in the interior). After applying γ, this says that $\gamma(z_0)$ must be exterior to the disks with the same labels in Figure 4.23.

The quantity

$$\frac{1}{C} = \frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} + \frac{1}{2\sqrt{2}}$$

is the diameter of the union of the disk C (respectively U) with the disk bounded by the black circle in Figure 4.23. This is the radius of the smallest circle centered in the shaded region whose interior avoids the labeled disks (i.e. forcing $\gamma(z_0)$ to be in the appropriate neighborhood of infinity).

$\square$

4.6 Antiprisms and questions

The construction and examples of the previous sections raise some questions. For instance:

- Given a two-colorable polyhedron, is there a natural field of definition for the associated reflection group? If so, what is it?
- Does the associated continued fraction algorithm have any arithmetic significance? If so, what is it?
- The three main examples of this chapter (ideal triangle, octahedron, cubeoctahedron) are all 2-congruence subgroups. Does this have any significance?

For example, some circles defining reflection groups for regular antiprisms are shown in Figure 4.24. The circles have (center, radius) equal to (where the antiprism consists of two n-gons and $2n$ triangles, and $1 \leq k \leq n$):

$$(0,1), \ \left(\frac{\zeta_n^k}{1+\sin(\pi/n)}, \frac{\sin(\pi/n)}{1+\sin(\pi/n)}\right) \text{ red circles,}$$

$$\left(0, \frac{\cos(\pi/n)}{1+\sin(\pi/n)}\right), \ \left(\frac{\zeta_{2n}^k}{\cos(\pi/n)}, \tan(\pi/n)\right) \text{ blue circles.}$$

These coordinates are somewhat arbitrary (i.e. whatever I could conveniently write down), but they are defined over the cyclotomic field $\mathbb{Q}(\zeta_{2n})$.

However, the $n = 3$ case is the ideal right-angled octahedron, and the $n = 4$ case is the ideal right-angled cubeoctahedron cut in half. This is interesting because we know the octahedron can be

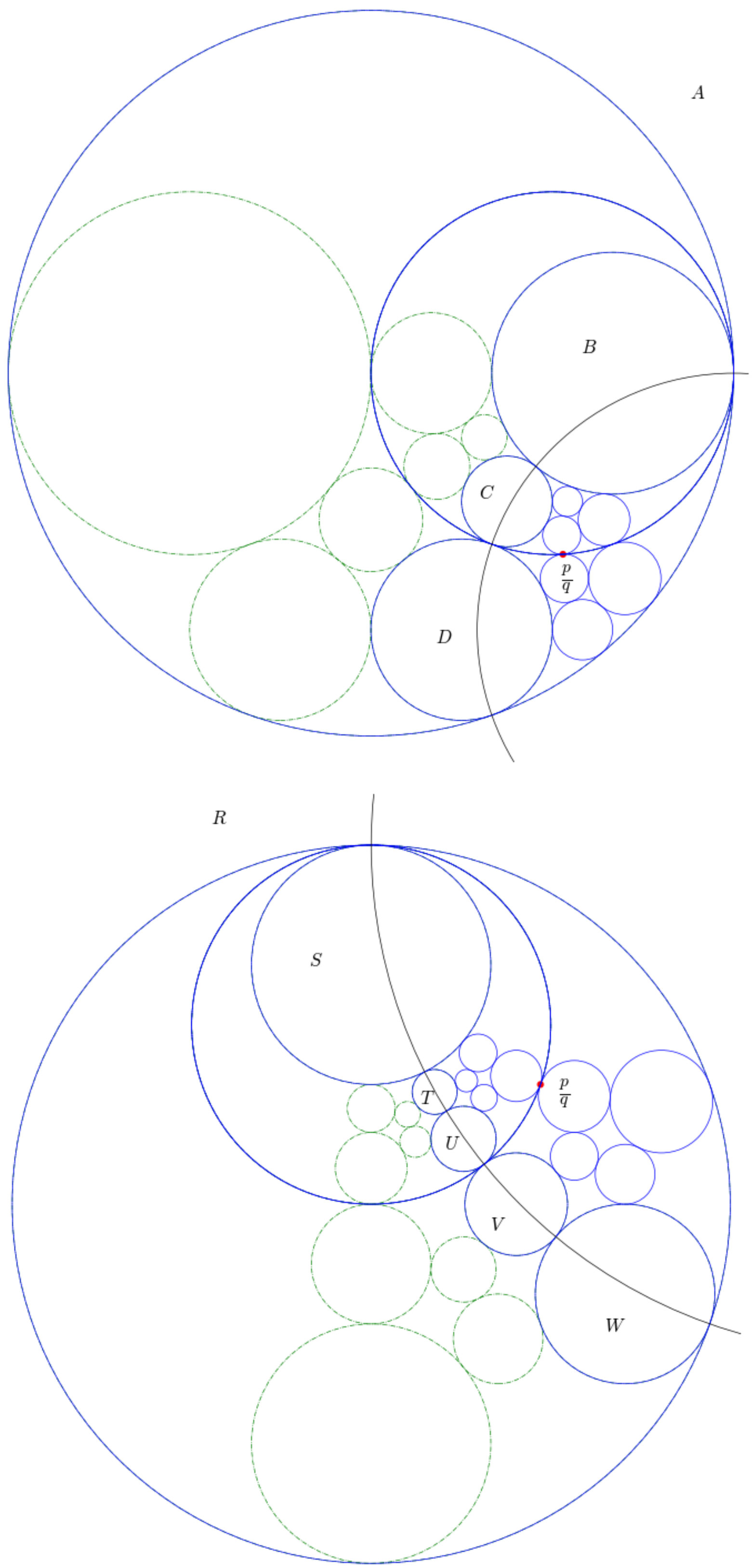

Figure 4.22: A rational number p/q appearing as an approximation to z_0 for the first time with respect to T_A (top) or T_B (bottom). In particular z_0 must be outside the labeled disks (the interiors of A and R include infinity).

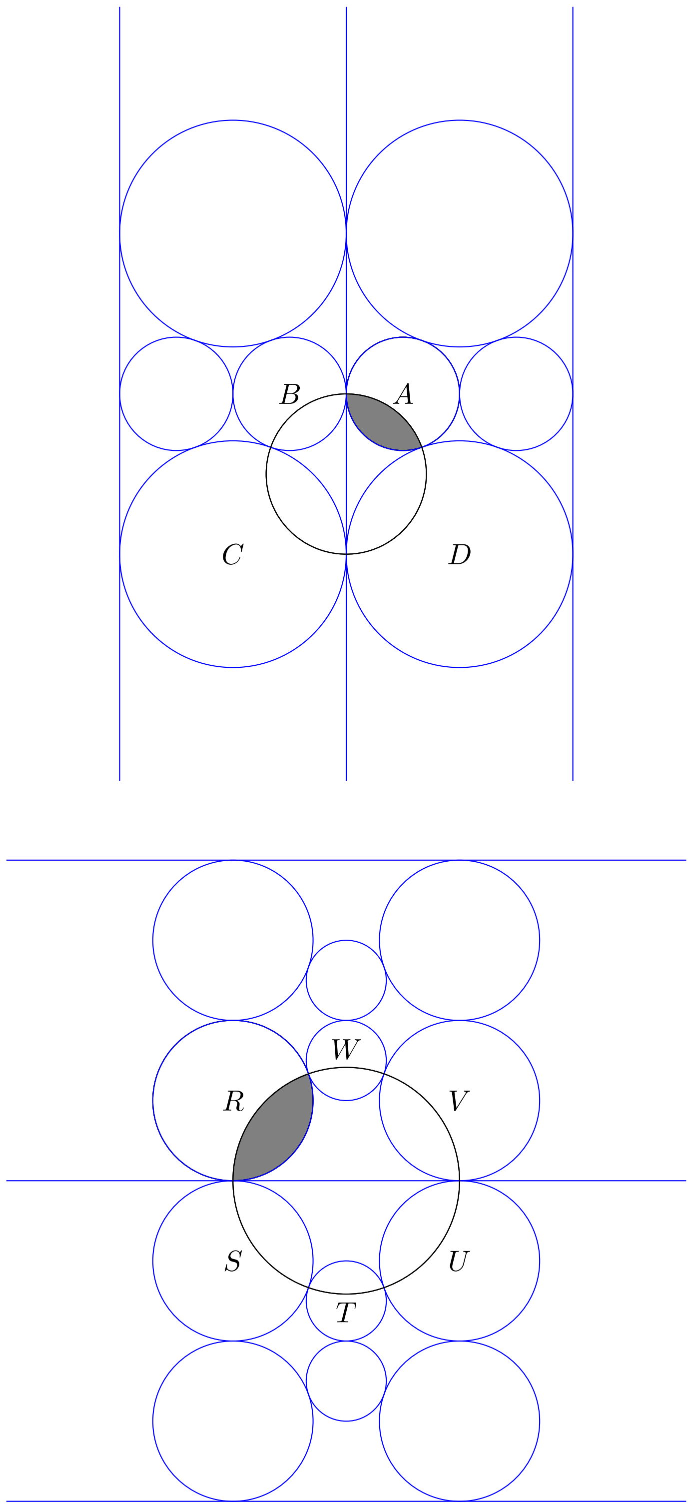

Figure 4.23: Result of applying γ to the configurations of Figure 4.22, with $\gamma(\infty)$ in the shaded region. If p/q were an approximation to z_0, then z_0 must be outside of the labeled disks by definition of the algorithms associated to T_A and T_B.

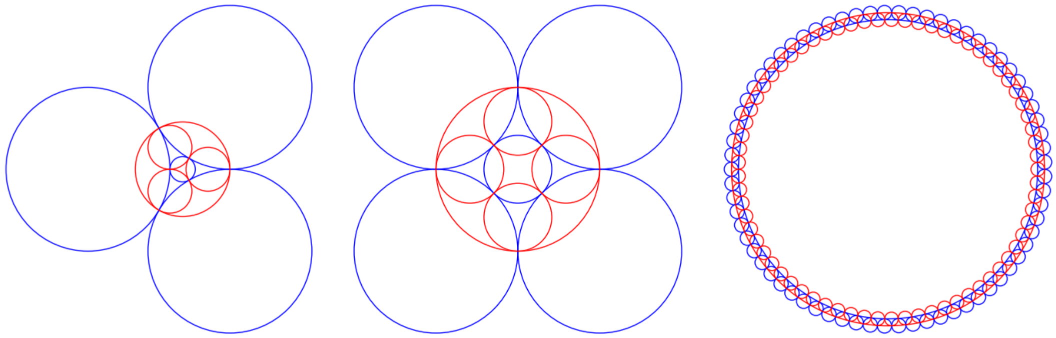

Figure 4.24: Antiprism reflection generators, $n = 3, 4, 69$.

defined over $\mathbb{Q}(\sqrt{-1})$ and the cubeoctahedron can be defined over $\mathbb{Q}(\sqrt{-2})$ where the orbit of the vertices gives the enitire field in a systematic way. So, somehow, the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ can be "embedded" into these cyclotomic fields. At the very least this gives an injection of sets, but perhaps it preserves some other structure, such as the heights of the points.

Finally, we note a general construction for constructing ideal polyhedral cell structures on $\mathcal{H}^3$ that are invariant under the Bianchi groups $PSL_2(\mathcal{O}_K)$, K imaginary quadratic (cf. [Yas10] for the construction, applications, and some data). For the five Euclidean imaginary quadratic fields, this complex consists of one orbit of a single polyhedron (the octahedron, cubeoctahedron, tetrahedon, triangular prism, and truncated tetrahedron for $-d = 1, 2, 3, 7, 11$ and $K = \mathbb{Q}(\sqrt{-d})$), which is right-angled in the first two examples.

Chapter 5

Characterizations and examples of badly approximable numbers

5.1 Introduction

This chapter gives characterizations and examples of badly approximable vectors in various settings, starting with a discussion of the prototypical one-dimensional case in the next section. Much of this chapter comes from [Hin18a] and [Hin18b]. Here is an outline, followed by a general discussion.

- (5.2) Badly approximable real numbers $\xi = [a_0; a_1, a_2, \ldots]$ over $\mathbb{Q}$ are characterized by bounded partial quotients $|a_n| \leq M$, or (more or less equivalently) bounded geodesic trajectories on the modular surface, $\{\xi + e^{-t}i : t \geq 0\} \bmod SL_2(\mathbb{Z})$. Obvious examples of badly approximable numbers are given by quadratic irrationals, whose partial quotients are eventually periodic, and whose associated trajectory is asymptotic to a closed (periodic) geodesic on the modular surface, namely the projection of the geodesic with ideal endpoints ξ, ξ' (prime denoting Galois conjugate). There is also another, simpler, argument of Liouville (concerning rational approximations to algebraic numbers) showing that quadratic irrationalities are badly approximable.

- (5.3) In this section we show that complex numbers badly approximable over the Euclidean imaginary quadratic fields are characterized by bounded partial quotients in their nearest integer continued fraction expansion.

- (5.4) Here we show that badly approximable complex numbers over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$

are characterized by orbits bounded away from fixed points under the octahedral and cubeoctahedral continued fraction algorithms of chapter 4.

- (5.5) Here we state a version of Mahler's criterion and the Dani correspondence (cf. chapter 2) characterizing badly approximable vectors. See [EGL16] for proofs, or 2.4 for details in the imaginary quadratic case.

- (5.6) In this section we give specific examples of badly approximable vectors associated to compact totally geodesic subspaces of the "modular" spaces

$$\Gamma\backslash G/K = SL_2(\mathcal{O}_F)\backslash(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s.$$

  These have simple number theoretic descriptions in terms of totally indefinite anisotropic rational binary quadratic and Hermitian forms. These examples are natural generalizations of real quadratic irrationals from the one-dimensional situation of 5.2.

- (5.7) While the previous section discussed the quadratic/Hermitian examples from the point of view of the Dani correspondence, here we give arguments using nearest integer continued fractions when F is a Euclidean imaginary quadratic field.

- (5.8) Here we discuss the Hermitian examples from the point of view of right-angled continued fractions over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$.

- (5.9) This section gives a Liouville-type argument showing that our quadratic/Hermitian examples are badly approximable. In the imaginary quadratic case, we give estimates of the approximation constants for badly approximable Hermitian zeros.

- (5.10) Finally, we point out that among the Hermitian examples there are many *algebraic* vectors and give a characterization of these, with an emphasis on the imaginary quadratic case (if only for the pictures).

We end this introduction with a general discussion and some context. A number field F of degree $r + 2s$ embeds naturally in the product of its Archimedean completions $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$.

Given a vector $\mathbf{z} = (z_1, \ldots, z_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^s$, one can ask how well $\mathbf{z}$ can be approximated by elements of F. Following [EGL16] and [KL16], we will measure the quality of approximation by

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\},\ p/q \in F,\ p, q \in \mathcal{O}_F,$$

where p_i and q_i are the images of p and q under $r+s$ inequivalent embeddings $\sigma_i : F \to \mathbb{C}$ and $|\cdot|$ is the usual absolute value in $\mathbb{R}$ or $\mathbb{C}$. The measure above is meaningful in the sense that all irrational vectors have infinitely many "good" approximations as demonstrated by the following Dirichlet-type theorem.

Theorem 5.1.1 (cf. [Quê91], Theorem 1)**.** *There is a constant C depending only on F such that for any $\mathbf{z} \notin F$*

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \leq C$$

has infinitely many solutions $p/q \in F$.

In what follows, we will give some explicit examples showing that the above theorem fails if the constant is decreased, i.e. there are *badly approximable* vectors, $\mathbf{z}$ such that there exists $C' > 0$ with

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \geq C'$$

for all $p/q \in F$. Our examples come from "obvious" compact totally geodesic subspaces of

$$SL_2(\mathcal{O}_F)\backslash(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s,$$

where $\mathcal{H}^n$ is hyperbolic n-space. Namely, these examples are associated to totally indefinite anisotropic F-rational binary quadratic forms over any number field (Proposition 5.6.1) and totally indefinite anisotropic F-rational binary Hermitian forms over CM fields (Proposition 5.6.3). Among the examples are algebraic vectors, i.e. vectors whose entries generate a non-trivial finite extension of F, including non-quadratic vectors in the CM case (Corollary 5.10.1). This is interesting in light of the following variation on Roth's theorem (which can be deduced from the Subspace Theorem for number fields).

Theorem 5.1.2 (cf. [Sch75b], Theorem 3). *Suppose $\mathbf{z} \notin F$ has algebraic coordinates. Then for all $\epsilon > 0$, there exists a constant $C' > 0$ depending on $\mathbf{z}$ and ϵ such that*

$$\max_i\{|q_i|\}^{1+\epsilon} \max_i\{|q_i z_i - p_i|\} \geq C'$$

for all $p/q \in F$.

The "linear forms" notion of badly approximable defined above implies

$$\max_i\{|z_i - p_i/q_i|\} \geq \frac{C'}{\max_i\{|q_i|^2\}} \text{ for all } p/q \in F,$$

which is perhaps the first notion of badly approximable that comes to mind. The two notions are equivalent when F has only one infinite place (i.e. $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$) since the absolute value is multiplicative. (The notions are also equivalent for real quadratic and complex quartic F, [KL16] Proposition A.2.) However, in larger number fields it seems that some choice must be made and the linear choice appears naturally in the proof of Theorem 5.5.1.

Simultaneous approximation in this sense seems natural and has been explored by various authors, e.g. [EGL16], [Hat07], [KL16], [Quê91], [Sch75b],[Bur92]. Among known facts, we note that the set of badly approximable vectors has Lebesgue measure zero, full Hausdorff dimension, and is even "winning" when restricted to curves and various fractals in $\mathbb{R}^r \times \mathbb{C}^s$ ([EGL16], [KL16], [ESK10]).

5.2 Simple continued fractions, the modular surface, and quadratic irrationalities

Below are two characterizations of badly approximable real numbers. One uses continued fractions and the other bounded geodesic trajectories on the modular surface.

Theorem 5.2.1. *An irrational real number $\xi = [a_0, a_1, a_2, \ldots]$ is badly approximable if and only if its partial quotients a_n are bounded.*

Proof. If ξ is badly approximable, $|\xi - p/q| \geq C'/q^2$, then in particular

$$\frac{C'}{q_n^2} \leq |\xi - p_n/q_n| = \frac{1}{q_n^2([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1])} \leq \frac{1}{q_n^2 a_{n+1}},$$
$$a_{n+1} \leq 1/C'.$$

Conversely, if the partial quotients are bounded, $\sup_n\{a_n\} \leq M$, then for any p/q with $0 < q \leq q_n$

$$|\xi - p/q| \geq |\xi - p_n/q_n| = \frac{1}{q_n^2(q_{n+1}/q_n + \xi_{n+1})}$$
$$= \frac{1}{q_n^2([0; a_{n+2,\ldots}] + [a_{n+1}; a_n, \ldots, a_1])} \geq \frac{1}{q_n^2\,(a_{n+1}+2)} \geq \frac{1}{q_n^2(M+2)},$$

using the fact that the convergents p_n/q_n are the *best* approximations for the first inequality. □

Theorem 5.2.2. *The number ξ is badly approximable if and only if the trajectory*

$$\Omega_\xi = \left\{ SL_2(\mathbb{Z}) \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$$

is bounded (precompact).

Proof. Suppose ξ is badly approximable with $|q(q\xi + p)| \geq C'$ for all p, q. If

$$\left\| (q\ p) \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right\|_\infty = \|(e^{-t}q, e^t(q\xi + p))\|_\infty < \sqrt{C'},$$

then taking the product of the coordinates gives a contradiction. Hence Ω_ξ is precompact by Mahler's criterion.

Conversely, if ξ is not badly approximable, there exist sequences a_n, b_n such that $|b_n(b_n\xi + a_n)| \leq 1/n^2$. If t_n is such that $e^{-t_n}|b_n| \leq 1/n$, then

$$\|(e^{-t_n}b_n, e^{t_n}(b_n\xi + a_n))\|_\infty \leq 1/n,$$

and the trajectory Ω_ξ is not bounded (Mahler's criterion again). □

The obvious (and conjecturally only) badly approximable real algebraic numbers are quadratic irrationals. Below we present three proofs of this fact, all of which we will revisit in the sequel.

Lemma 5.2.3 (Liouville, 1844)**.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left|\xi - \frac{h}{k}\right| > \frac{A}{k^n}.$$

(In particular, real quadratics are badly approximable.)

Proof. Suppose $p(x) \in \mathbb{Z}[x]$ is irreducible of degree n with $p(\xi) = 0$. Then

$$p(\xi) - p(h/k) = (\xi - h/k)p'(\alpha)$$

for some α between ξ and h/k by the mean value theorem. The left hand side is a non-zero rational number ($p(\xi) = 0$ and p is irreducible so h/k is not a root) with denominator less than k^n so that we get

$$\frac{1}{k^n} \leq \left|\xi - \frac{h}{k}\right| \sup\{p'(x) : x \in (\xi - 1, \xi + 1)\}.$$

$\square$

Theorem 5.2.4 (Lagrange, 1770)**.** *The partial quotients of a quadratic irrational are eventually periodic (hence bounded).*

Proof. (Cf. [Khi97] Theorem 28.) Let

$$Q(x, y) = Ax^2 + Bxy + Cy^2 = (x\ y)\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$
$$A, B, C \in \mathbb{Z},\ \Delta(Q) = AC - B^2/4 < 0,$$

be an indefinite integral binary quadratic form, and $Z(Q)$ its zero set (i.e. two points in $P^1(\mathbb{R})$). $GL_2(\mathbb{Z})$ acts by change of variable on Q, by Möbius transformations on $Z(Q)$, and the map $Q \mapsto Z(Q)$ is $GL_2(\mathbb{Z})$-equivariant:

$$Z(Q^g) = Z(g^t Q g) = g^{-1} \cdot Z(Q).$$

Note that the action on forms preserves the determinant Δ. For $\xi \in \mathbb{R}$, $\xi = [a_0; a_1, \ldots]$, let

$$g_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

so that $T^n(\xi_0) = 1/g_n^{-1}\xi$. If $Q(\xi, 1) = 0$ for some indefinite integral binary quadratic form, let $Q_n = Q^{g_n} = A_n x^2 + B_n xy + C_n y^2$, and note that $Q_n(1, \xi_n) = 0$. We claim that the collection $\{Q_n : n \geq 0\}$ is finite. The form Q_n is given by

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

so that its coefficents satisfy

$$\begin{aligned} A_n &= Ap_n^2 + Bp_nq_n + C_nq_n^2, \\ B_n &= 2Ap_nq_n + B(p_nq_{n-1} + p_{n-1}q_n) + Cq_nq_{n-1}, \\ C_n &= Ap_{n-1}^2 + Bp_{n-1}q_{n-1} + Cq_{n-1}^2 = A_{n_1}. \end{aligned}$$

Because $|\xi - p_n/q_n| \leq 1/q_n^2$, we have $q_n\xi - p_n = \delta_n/q_n$ for some δ_n with $|\delta_n| \leq 1$. Substituting this into the above gives

$$\begin{aligned} A_n &= A(q_n\xi - \delta_n/q_n)^2 + B(q_n\xi - \delta_n/q_n)q_n + Cq_n^2 \\ &= (A\xi^2 + B\xi + C)q_n + A(\delta_n^2/q_n^2 - 2\xi\delta_n) - B\delta_n \\ &= A(\delta_n^2/q_n^2 - 2\xi\delta_n) - B\delta_n, \\ |A_n| &\leq |2A\xi| + |A| + |B|. \end{aligned}$$

Hence A_n, $C_n = A_{n-1}$, and $B_n = \pm\sqrt{A_nC_n - 4\Delta}$ are all bounded in terms of A, B, C, and ξ. This proves the claim and shows that the sequence ξ_n and a_n are eventually periodic. $\square$

Theorem 5.2.5. *Real quadratic irrationals are badly approximable (via the Dani correspondence).*

Proof. The closed (compact/periodic) geodesics on the modular surface are the projections of the geodesics in $\mathcal{H}^2$ joining conjugate real quadratic irrationals (upper half-plane model). In one direction, the fixed points $z = \frac{az+b}{cz+d}$ of a hyperbolic element of $SL_2(\mathbb{Z})$ are real quadratic irrationals. In the other, the stabilizer of the quadratic form associated to the minimal polynomial of a quadratic irrational is generated by a hyperbolic element (which can be written explicitly in terms of a fundamental solution to Pell's equation, cf. [Lan58] Theorem 202).

If $Q(\xi, 1) = 0$ for an integral form, then the geodesic trajectory Ω_ξ is asymptotic to the closed geodesic joining ξ, $\bar{\xi}$. Therefore Ω_ξ is bounded and ξ is badly approximable. $\square$

5.3 Bounded partial quotients (nearest integer)

Badly approximable complex numbers (over the Euclidean imaginary quadratic fields) can also be characterized by boundedness of their partial quotients.

Theorem 5.3.1. *A number $z \in \mathbb{C} \setminus K$ is badly approximable if and only if its partial quotients a_n are bounded (if and only if the remainders z_n are bounded away from zero). In particular, if $|a_n| \leq \beta$ for all n and $p/q \in K$, then $|z - p/q| \geq C'/|q|^2$ where*

$$C' = \frac{1}{\alpha(\beta+1)(\beta+\rho+1)}$$

and $\rho = \rho_d$ is as in 3.3.

Proof. If z is badly approximable, then there is a $C' > 0$ such that for each convergent p_n/q_n to z, the disk $|w - p_n/q_n| \leq C'/|q_n|^2$ does not contain z. Mapping p_n/q_n to ∞ via g_n^{-1}, where

$$g_n = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

maps the disk $|w - p_n/q_n| \leq C'/|q_n|^2$ to the region $|w + q_{n-1}/q_n| \geq 1/C'$, centered at $g_n^{-1}(\infty) = -q_{n-1}/q_n$. Because $g_n^{-1}(z)$ is inside the disk of radius $1/C'$ centered at $-q_{n-1}/q_n$ and $|-q_{n-1}/q_n| < 1$, we have

$$a_{n+1} + z_{n+1} = 1/z_n = g_n^{-1}(z),$$

$$|a_{n+1}| \leq |z_{n+1}| + |g_n^{-1}(z)| \leq \rho + 1 + 1/C'.$$

Hence a_{n+1} is bounded. See Figure 5.1 below for an illustration.

By Lemma 3.3.3, for z and p/q with $|q_{n-1}| < |q| \le |q_n|$ we have

$$\begin{aligned}
\left|z - \frac{p_n}{q_n}\right| \le \alpha \left|z - \frac{p}{q}\right| \left|\frac{q}{q_n}\right| &\le \alpha \left|z - \frac{p}{q}\right| \frac{|q|^2}{|q_n|^2} \left|\frac{q_n}{q_{n-1}}\right| \\
&= \alpha \left|z - \frac{p}{q}\right| \frac{|q|^2}{|q_n|^2} \left|a_n + \frac{q_{n-2}}{q_{n-1}}\right| \\
&\le \alpha \left|z - \frac{p}{q}\right| \frac{|q|^2}{|q_n|^2}(|a_n| + 1), \\
|q_n|^2 \left|z - \frac{p_n}{q_n}\right| &\le \alpha(|a_n| + 1)|q|^2 \left|z - \frac{p}{q}\right|.
\end{aligned}$$

This shows that if z has bounded partial quotients, then z is badly approximable if and only if it is badly approximable *by its convergents*. For approximation by convergents, we have

$$\begin{aligned}
\left|z - \frac{p_n}{q_n}\right| &= \frac{1}{|q_n|^2 |z_n^{-1} + q_{n-1}/q_n|} \\
&= \frac{1}{|q_n|^2 |a_{n+1} + z_{n+1} + q_{n-1}/q_n|} \ge \frac{1}{|q_n|^2(|a_{n+1}| + \rho + 1)},
\end{aligned}$$

showing that if the partial quotients of z are bounded, then z is badly approximable by convergents and therefore badly approximable. For an approximation constant, the above discussion gives $|z - p/q| \ge C'/|q|^2$ for any $p/q \in K$ where

$$C' = \frac{1}{\alpha(\beta + 1)(\beta + \rho + 1)}$$

and β is an upper bound for the $|a_n|$. □

5.4 Orbits bounded away from fixed points (right-angled)

Below is a characterization of badly approximable numbers in terms of its orbit under the octahedral and cubeoctahedral continued fraction maps of chapter 4.

Theorem 5.4.1. *A number $z_0 \in \mathbb{C} \setminus \mathbb{Q}(i)$ is badly approximable if and only if its T-orbit is bounded away from the fixed points (six vertices of the octahedron).*

Proof. If z_0 is badly approximable, let p/q be a convergent to z_0, without loss of generality in the orbit of ∞. If $g \in \Gamma$ is the element corresponding to repeated application of T, $g(p/q) = \infty$, then

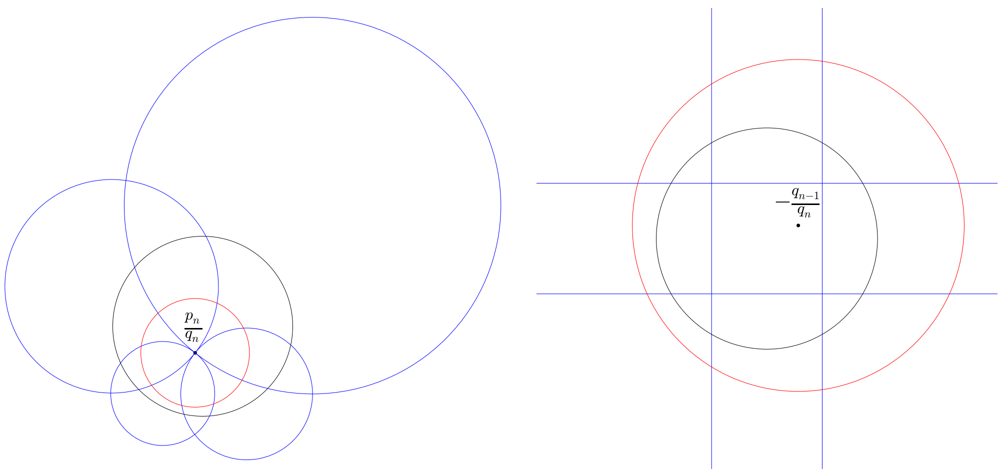

Figure 5.1: Over $\mathbb{Q}(\sqrt{-1})$, we have the points $p_n/q_n = g_n(\infty)$ and $-q_{n-1}/q_n = g_n^{-1}(\infty)$, along with the unit circle and its image under g_n (black), circles of radius $1/C'$ and $C'/|q_n|^2$ (red), and the lines defining V and their images under g_n (blue).

the disk $|z - p/q| < C/|q|^2$ (not containing z_0) gets mapped to the exterior of the disk centered at $g(\infty)$ (which is in one of the regions adjacent to $\frac{1}{1-i}$) of radius $1/C$. Hence $g(z_0)$ is bounded. Similarly (or applying an automorphism) for the other fixed points. If z_0 is not badly approximable (say by convergents in the orbit of ∞), then the above argument shows that $g(z_0)$ is contained in the exterior of arbitrarily large circles centered near $\frac{1}{1-i}$. Again, ∞ is not special in this argument, just convenient. We can limit ourselves to approximation by convergents since they exhaust all rationals with $|z - p/q| < \frac{1}{(1+q/\sqrt{2})|q|^2}$. □

An argument in the same vein as the previous proof furnishes a similar statement for the cubeoctahedral continued fractions.

Theorem 5.4.2. *A number $z_0 \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-2})$ is badly approximable if and only if its T-orbit is bounded away from the fixed points (12 vertices of the cubeoctahedron).*

5.5 Bounded geodesic trajectories (Dani correspondence)

The previous two sections described characterizations of badly approximable numbers over in terms of continued fractions, but we can still give a characterization of badly approximable vectors via the Dani correspondence even when continued fractions aren't available.

Theorem 5.5.1 ([EGL16], Proposition 3.1)**.** *The vector $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^r \times \mathbb{C}^s$ is badly approximable over F if and only if the geodesic trajectory*

$$\Gamma \cdot \Omega_{\mathbf{z}} \cdot K \subseteq \Gamma \backslash (\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$$

is bounded, where

$$\Omega_{\mathbf{z}} = \left\{ \left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}, \ldots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right) : 0 \le t \in \mathbb{R} \right\}.$$

This follows in a straight-forward fashion from the following version of Mahler's compactness criterion, stated here for SL_2.

Theorem 5.5.2 ([EGL16], Theorem 2.2)**.** *A subset $\Gamma \cdot \Omega \subseteq \Gamma\backslash SL_2(F \otimes \mathbb{R})$ is precompact if and only if there exists $\epsilon > 0$ such that*

$$\max\{\max_i\{|z_i|\}, \max_i\{|w_i|\}\} \geq \epsilon, \ (\mathbf{z}, \mathbf{w}) = (q, p)\omega,$$

for all $(0,0) \neq (q,p) \in \mathcal{O}_F^2$ and $\omega \in \Omega$. In other words, the two-dimensional $\mathcal{O}_F$-modules in $(F \otimes \mathbb{R})^2$ spanned by the rows of $\omega \in \Omega$ do not contain arbitrarily short vectors.

5.6 Examples (zeros of totally indefinite anisotropic rational binary quadratic and Hermitian forms)

In this section we give explicit examples of badly approximable vectors, namely zeros of totally indefinite F-rational binary quadratic and Hermitian forms.

5.6.1 Totally indefinite binary quadratic forms

As above, let F be a number field, $F \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, $\Gamma = SL_2(\mathcal{O}_F)$, and let

$$Q(x,y) = (x \ y) \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax^2 + Bxy + Cy^2, \ A, B, C \in F$$

be an F-rational binary quadratic form with determinant $\Delta(Q) = AC - B^2/4$. We say Q is *totally indefinite* if $\sigma(\Delta) < 0$ for all real embeddings $\sigma : F \to \mathbb{R}$ and that Q is *anisotropic* if it has no non-trivial zeros in F^2. Note that Q is anisotropic if and only if $-\Delta(Q)$ is not a square in F. Let Q_i be the form obtained by applying σ_i to the coefficients of Q, denote by $Z_i(Q)$ the zero set of Q_i

$$Z_i(Q) = \{[z : w] \in P^1(\mathbb{C}) \text{ or } P^1(\mathbb{R}) : Q_i(z,w) = 0\},$$

and let $Z(Q) = \prod_i Z_i(Q)$ (a finite set of cardinality 2^n for totally indefinite Q).

The group Γ acts on binary quadratic forms by change of variable

$$Q^g(x,y) = (g^t Q g)(x,y) = Q(ax + by, cx + dy)$$

and also on $P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$ diagonally by linear fractional transformations

$$g \cdot ([z_1 : w_1], \ldots, [z_n : w_n]) = ([a_1 z_1 + b_1 w_1 : c_1 z_1 + d_1 w_1], \ldots, [a_n z_n + b_n w_n : c_n z_n + d_n w_n]),$$

where $a_i = \sigma_i(a)$ and similary for b_i, c_i, and d_i. These actions are compatible in the sense that $g^{-1} \cdot Z(Q) = Z(Q^g)$. Without further remark, we identify $\mathbb{R}^r \times \mathbb{C}^s$ with a subset of $P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$ via $(z_i)_i \mapsto ([z_i : 1])_i$.

5.6.2 Totally indefinite binary Hermitian forms over CM fields

Let F be a CM field (an imaginary quadratic extension of a totally real field E) of degree $2n$ with ring of integers $\mathcal{O}_F$ and let H be an F-rational binary Hermitian form

$$H(z,w) = (\bar{z}\ \bar{w}) \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = Az\bar{z} + \overline{B}z\bar{w} + B\bar{z}w + Cw\bar{w},\ A, C \in E,\ B \in F,$$

where the overline is "complex conjugation" (the non-trivial automorphism of F/E). Let H_i be the form obtained by applying σ_i to the coefficients of H (noting that σ_i commutes with complex conjugation). We say H is *totally indefinite* if $\sigma_i(\Delta) < 0$ for all i, where $\Delta = \det(H) = AC - B\overline{B}$. We say H is *anisotropic* if $H(p,q) \neq 0$ for $(p,q) \in F^2 \setminus \{(0,0)\}$. Note that H is anisotropic if and only if $-\Delta$ is not a relative norm, $-\Delta \notin N_E^F(F)$. Denote by $Z_i(H)$ the zero set of H_i,

$$Z_i(H) = \{([z:w] \in P^1(\mathbb{C}) : H_i(z,w) = 0\},$$

a circle in $P^1(\mathbb{C})$, and let $Z(H) = \prod_i Z_i(H)$. When H is totally indefinite, $Z(H) \cong (S^1)^s$ is an s-dimensional torus.

The group $\Gamma = SL_2(\mathcal{O}_F)$ acts on H by change of variable

$$H^g(z,w) = (\bar{g}^t H g)(z,w) = H(az+bw, cz+dw),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F),$$

and also on $P^1(\mathbb{C})^n$ diagonally by linear fractional transformations

$$g \cdot ([z_1 : w_1], \ldots, [z_n : w_n]) = ([a_1 z_1 + b_1 w_1 : c_1 z_1 + d_1 w_1], \ldots, [a_n z_n + b_n w_n : c_n z_n + d_n w_n]).$$

where $a_i = \sigma_i(a)$ and similarly for b_i, c_i, and d_i. These actions are compatible in the sense that $g^{-1} \cdot Z(H) = Z(H^g)$. As before, we include $\mathbb{C}^s \hookrightarrow P^1(\mathbb{C})^s$ via $(z_i)_i \mapsto ([z_i : 1])_i$.

5.6.3 Examples from quadatic forms

The following is a generalization of the fact that quadratic irrationals are badly approximable over $\mathbb{Q}$ ($r = 1$, $s = 0$), as discussed in 5.2 above. We should note that these examples can also be deduced from Theorem 6.4 of [Bur92] (with S the set of infinite places and $N = 1$).

Proposition 5.6.1. *Let Q be a totally indefinite anisotropic F-rational binary quadratic form over a number field F. Then any vector $\mathbf{z} \in Z(Q)$ is badly approximable over F.*

First we establish compactness of a subspace associated to Q. There are many references with discussions of compactness for anisotropic arithmetic quotients, e.g. [PR94], [Rag72], and [Mor15].

Lemma 5.6.2. *Let Q be a totally indefinite F-rational anisotropic binary quadratic form, and let L_i be the line in $\mathcal{H}^2$ or $\mathcal{H}^3$ with endpoints $Z(Q_i)$. Then $\pi(\prod_i L_i)$ is compact in $\Gamma\backslash(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$, where $\pi : (\mathcal{H}^2)^r \times (\mathcal{H}^3)^s \to \Gamma\backslash(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$ is the quotient map.*

Proof of Lemma 5.6.2. Without loss of generality, suppose Q has integral coefficients, $A, B, C \in \mathcal{O}_F$. Compactness of

$$SO(Q, \mathcal{O}_F)\backslash SO(Q, F \otimes \mathbb{R}) \subseteq \Gamma\backslash SL_2(F \otimes \mathbb{R}),$$

is a consequence of Mahler's compactness criterion as follows. For $g \in SO(Q, F \otimes \mathbb{R})$ and any $(0, 0) \neq (\alpha, \beta) \in \mathcal{O}_F^2$, the quantity $\max_i\{|\sigma_i(Q^g(\alpha, \beta))|\}$ is bounded away from zero because

$$0 \neq Q^g(\alpha, \beta) = Q(\alpha, \beta) \in \mathcal{O}_F,$$

and $\mathcal{O}_F$ is discrete in $F \otimes \mathbb{R}$. By Mahler's criterion and continuity of Q viewed as a function $(\mathbb{R}^r \times \mathbb{C}^s)^2 \to \mathbb{R}^r \times \mathbb{C}^s$, $SO(Q, \mathcal{O}_F)\backslash SO(Q, F \otimes \mathbb{R})$ is precompact.[1] The inclusion above is a closed embedding, hence its image is compact.

To get the result in the locally symmetric space, note that

$$\pi(\prod_i L_i) = \Gamma \cdot SO(Q, F \otimes \mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K,$$

[1] Due to choices of left/right actions, this technically shows that $SO(F \otimes \mathbb{R})/SO(\mathcal{O}_F)$ has compact closure in $SL_2(F \otimes \mathbb{R})/\Gamma$. However the left and right coset spaces are homeomorphic.

where $g \in SL_2(F\otimes\mathbb{R})$ is any element such that $gK \in \prod_i L_i$, and that $\Gamma\backslash G \to \Gamma\backslash G/K$ is proper. $\square$

For a concrete example, the four vectors

$$\left(\pm\sqrt{2-\sqrt{2}}, \pm\sqrt{2+\sqrt{2}}\right) \in \mathbb{R}^2$$

are badly approximable over $\mathbb{Q}(\sqrt{2})$ as they are roots of the totally indefinite anisotropic binary quadratic form $Q(x,y) = x^2 - (2-\sqrt{2})y^2$ (anisotropic since $2-\sqrt{2}$ is not a square in $\mathbb{Q}(\sqrt{2})$).

5.6.4 Examples from Hermitian forms

The following is a generalization of the fact that zeros of anisotropic binary Hermitian forms are badly approximable over imaginary quadratic fields ($r = 0$, $s = 1$). As in the case of quadratic irrationals over $\mathbb{Q}$, this can be demonstrated with continued fractions when the imaginary quadratic field is Euclidean, $F = \mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. Details for the imaginary quadratic case can be found in [Hin18a] or further on in this chapter.

Proposition 5.6.3. *If $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a zero of the totally indefinite anisotropic F-rational binary Hermitian form H, i.e. $\mathbf{z} \in Z(H)$, then $\mathbf{z}$ is badly approximable.*

As before, we first establish compactness of a subspace associated to H (once again, cf. [**?**], [Rag72], or [Mor15]).

Lemma 5.6.4. *If H is a totally indefinite anisotropic F-rational binary Hermitian form, then $\pi\left(\prod_i P_i\right)$ is compact in $\Gamma\backslash(\mathcal{H}^3)^n$ where P_i is the geodesic plane in the ith copy of $\mathcal{H}^3$ whose boundary at infinity is the zero set $Z_i(H)$ and $\pi : (\mathcal{H}^3)^n \to \Gamma\backslash(\mathcal{H}^3)^n$ is the quotient map.*

Proof of Lemma 5.6.4. Without loss of generality, suppose H has integral coefficients, $A, C \in \mathcal{O}_F$, $B \in \mathcal{O}_E$. Compactness of

$$SU(H, \mathcal{O}_F)\backslash SU(H, F\otimes\mathbb{R}) \subseteq \Gamma\backslash SL_2(F\otimes\mathbb{R}),$$

follows from Mahler's compactness criterion as follows. For $g \in SU(H, F\otimes\mathbb{R})$ and any $(0,0) \neq (\alpha,\beta) \in \mathcal{O}_F^2$, the quantity $\max_i\{|\sigma_i(H^g(\alpha,\beta))|\}$ is bounded away from zero because

$$0 \neq H^g(\alpha,\beta) = H(\alpha,\beta) \in \mathcal{O}_E,$$

and $\mathcal{O}_E$ is discrete in $F \otimes \mathbb{R}$. By Mahler's criterion and continuity of H viewed as a function $(\mathbb{C}^s)^2 \to \mathbb{C}^s$, $SU(H, \mathcal{O}_F)\backslash SU(H, F\otimes\mathbb{R})$ is precompact.[2] The inclusion above is a closed embedding, hence its image is compact.

To get the result in the locally symmetric space, note that $\pi(\prod_i P_i) = \Gamma \cdot SU(H, F\otimes\mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K$ where $g \in SL_2(F \otimes \mathbb{R})$ is any element such that $gK \in \prod_i P_i$, and that $\Gamma\backslash G \to \Gamma\backslash G/K$ is proper. $\square$

For a concrete example, the entire torus

$$\{(\sqrt{3}\cos s + i\sqrt{3}\sin s, \sqrt{3}\cos t + i\sqrt{3}\sin t) : s, t \in [0, 2\pi)\} \subseteq \mathbb{C}^2$$

is badly approximable over $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$ as it is the zero set of the totally indefinite anisotropic $H(z, w) = |z|^2 - 3|w|^2$ (anisotropic since 3 is not a relative norm).

5.7 Examples via nearest integer continued fractions

In this section we focus on producing z with bounded partial quotients, extending the results of [BG12] to all of the Euclidean imaginary quadratic fields $K_d = \mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. We will show that there are countably many circles in the complex plane all of whose points have bounded partial quotients in their nearest integer continued fractions.

We will consider equivalence classes of indefinite integral binary Hermitian forms. A *binary Hermitian form* $H(z, w)$ is a function of the form

$$\begin{aligned} H(z,w) &= (\overline{z}, \overline{w}) \begin{pmatrix} A & -B \\ -\overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \\ &= Az\overline{z} - B\overline{z}w - \overline{B}z\overline{w} + Cw\overline{w}, \ A, C \in \mathbb{R}, \ B \in \mathbb{C}. \end{aligned}$$

We denote by $\Delta(H)$ the determinant $\det(H) = AC - |B|^2$ of the Hermitian matrix defining H. The binary Hermitian form H is *integral* over K if the matrix entries of H are integers, i.e. $A, C \in \mathbb{Z}$ and $B \in \mathcal{O}$. The form is *indefinite* (takes on both positive and negative values) if and only if $\Delta(H) < 0$.

[2] Once again, due to choices of left/right actions, this technically shows that $SU(H, F \otimes \mathbb{R})/SU(H, \mathcal{O}_F)$ has compact closure in $SL_2(F \otimes \mathbb{R})/\Gamma$. However the left and right coset spaces are homeomorphic.

The zero set of an indefinite H on the Riemann sphere $P^1(\mathbb{C})$ is a circle (using homogeneous coordinates $[z:w]$ on the projective line)

$$Z(H) := \{[z:w] \in P^1(\mathbb{C}) : H(z,w) = 0\}$$

which is either a circle or a line in the chart $\mathbb{C}_z = \{[z:1] \in P^1(\mathbb{C})\}$

$$Z(H) \cap \mathbb{C}_z = \begin{cases} \{z : |z - B/A|^2 = -\Delta/A^2\} & \text{if } A \neq 0, \\ \{z : \mathrm{Re}(\overline{B}z) = C\} & \text{if } A = 0. \end{cases}$$

We will be interested in equivalence of forms over $GL_2(\mathcal{O})$, where $g \in GL_2(\mathbb{C})$ acts as a normalized linear change of variable on the left, ${}^gH = |\det(g)|(g^{-1})^* H g^{-1}$ (here $*$ denotes the conjugate transpose), and also with the Möbius action of $GL_2(\mathbb{C})$ on $P^1(\mathbb{C})$,

$$g \cdot [z:w] = [az + bw : cz + dw], \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We collect some easily verified facts in the following lemma.

Lemma 5.7.1. *The following hold for the action ${}^gH = |\det(g)|(g^{-1})^* H g^{-1}$, $g \in GL_2(\mathbb{C})$, on indefinite binary Hermitian forms.*

- *The action of $GL_2(\mathbb{C})$ is determinant preserving, i.e. $\Delta({}^gH) = \Delta(H)$.*
- *The map $H \mapsto Z(H)$ is $GL_2(\mathbb{C})$-equivariant (i.e. $g \cdot Z(H) = Z({}^gH)$).*

Furthermore, an integral form H is isotropic (i.e. $H(z,w) = 0$ for some $[z:w] \in P^1(K)$) if and only if $-\Delta(H)$ is in the image of the norm map $N^K_{\mathbb{Q}} : K \to \mathbb{Q}$.

Proof. For $g \in GL_2(\mathbb{C})$ we have

$$\det({}^gH) = |\det(g)|^2 \det((g^{-1})^* H g^{-1}) = \frac{|\det(g)|^2 \det(H)}{\det(g)\overline{\det(g)}} = \det(H).$$

The second bullet follows from

$$\begin{aligned} ((\bar{z}\ \bar{w})g^*)({}^gH)(g(z\ w)^t) &= |\det(g)|((\bar{z}\ \bar{w})g^*)((g^{-1})^* H g^{-1})g(z,w)^t \\ &= |\det(g)|H(z,w). \end{aligned}$$

Finally, the factorization (assuming $A \neq 0$ else $-\Delta = |B|^2$ is a norm and $H(1,0) = 0$)

$$AH(z,w) = |Az - Bw|^2 + \Delta|w|^2$$

shows that $-\Delta$ is a norm if and only if there are $z, w \in K$ not both zero with $H(z,w) = 0$. $\square$

Suppose H is an indefinite integral binary Hermitian form of determinant Δ and $z = [a_0; a_1, \ldots]$ satisfies $H(z,1) = 0$. Define $H_n = {}^{g_n^{-1}}H$, where

$$g_n^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -a_0 \end{pmatrix}, \; g_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

with notation

$$H_n = \begin{pmatrix} A_n & -B_n \\ -\overline{B_n} & C_n \end{pmatrix}.$$

In particular, we have

$$A_n = H(p_n, q_n),$$
$$C_n = H(p_{n-1}, q_{n-1}) = A_{n-1}.$$

Note that $H_n(1, z_n) = 0$ for all $n \geq 1$ because $g_n(1/z_n) = z$.

The main observation for us is the following theorem, which is essentially Theorem 4.1 of [BG12] generalized to the other Euclidean K and arbitrary integral binary Hermitian forms. One could follow the inductive geometric proof of [BG12], but we give an algebraic proof analogous to one demonstrating that real quadratic irrationals have eventually periodic simple continued fraction expansions, e.g. [Khi97], Theorem 28. In fact, we may as well note that the proof of Theorem 5.7.2 below applies *mutatis mutandis* to integral binary quadratic forms over K, showing that the continued fraction expansions of quadratic irrationals over K are eventually periodic as expected.

Theorem 5.7.2. *If $[z : 1]$ is a zero of an indefinite integral binary Hermitian form H, then the collection $\{H_n : n \geq 0\}$ is finite.*

Proof. In what follows, $\Delta = \Delta(H) = \Delta(H_n)$. The inequality

$$|z - p_n/q_n| \leq \frac{\kappa}{|q_n|^2}$$

allows us to write

$$p_n = q_n z + \frac{\gamma_n}{q_n}, \ |\gamma_n| \leq \kappa$$

where $\kappa = \sup_{z,n}\{|q_n||q_n z - p_n|\} < 2$ is the best constant from [Lak73] used in the proof of Lemma 3.3.3.

Substituting this into the formula for A_n above gives

$$\begin{aligned} A_n &= H(q_n z + \gamma_n/q_n, q_n) \\ &= |q_n|^2 H(z,1) + A\left(\overline{q_n z}\frac{\gamma_n}{q_n} + q_n z\frac{\overline{\gamma_n}}{\overline{q_n}} + \left|\frac{\gamma_n}{q_n}\right|^2\right) - \overline{B}\frac{\overline{q_n}}{q_n}\gamma_n - B\frac{q_n}{\overline{q_n}}\overline{\gamma_n} \\ &= A\left(\overline{q_n z}\frac{\gamma_n}{q_n} + q_n z\frac{\overline{\gamma_n}}{\overline{q_n}} + \left|\frac{\gamma_n}{q_n}\right|^2\right) - \overline{B}\frac{\overline{q_n}}{q_n}\gamma_n - B\frac{q_n}{\overline{q_n}}\overline{\gamma_n}, \end{aligned}$$

and

$$\begin{aligned} |A_n| &\leq |A|\kappa^2 + 2|B|\kappa + 2|A||z|\kappa \leq |A|\kappa^2 + 4|B|\kappa + 2\kappa\sqrt{-\Delta} \\ &\leq \max\{|A|, |A|\kappa^2 + 4|B|\kappa + 2\kappa\sqrt{-\Delta}\} =: \eta. \end{aligned}$$

(We take the max above so that the parameter η is useful for bounds on z_n, a_n when $n = 0$, cf. Corollary 5.7.4 below.) From this it follows that

$$\begin{aligned} |C_n| &= |A_{n-1}| \leq \eta, \\ |B_n| &= \sqrt{A_n C_n - \Delta} \leq \sqrt{\eta^2 - \Delta}, \end{aligned}$$

so that there are only finitely many possibilities for H_n. □

By requiring H to be anisotropic, we bound the finitely many circles $Z(H_n)$ away from zero and infinity, obtaining bounded partial quotients.

Corollary 5.7.3. *If $[z : 1]$ is a zero of an anisotropic indefinite integral binary Hermitian form H, then z has bounded partial quotients in its nearest integer continued fraction expansion (and is therefore badly approximable over K).*

A quantitative measure of the "hole" around zero (see Figures 5.5, 5.6) can be given in terms of the determinant $\Delta = \det(H)$ of the form, which in turn bounds the partial quotients and controls the approximation constant $|z - p/q| \geq C'/|q|^2$.

Corollary 5.7.4. *If $z \in Z(H)$ is a zero of the anisotropic integral indefinite binary Hermitian form H of determinant Δ, then the remainders z_n, $n \geq 0$, are bounded below by*

$$|z_n| \geq \frac{1}{\sqrt{-\Delta} + \sqrt{\eta^2 - \Delta}},$$

with partial quotients bounded above by

$$|a_n| \leq \rho + \sqrt{-\Delta} + \sqrt{\eta^2 - \Delta},$$

where η is as in the proof of Theorem 5.7.2. From these bounds, one can produce a $C'(H) > 0$ such that $|z - p/q| \geq C'/|q|^2$ for all $p/q \in K$ and $[z : 1] \in Z(H)$.

Proof. We use the notation of Theorem 5.7.2. Since $1/z_n$ lies on $Z(H_n) \cap \mathbb{C}_z$, which has radius $\sqrt{-\Delta}/|A_n|$ and center B_n/A_n, we have

$$\begin{aligned} \frac{1}{|z_n|} &\leq \left|\frac{B_n}{A_n}\right| + \frac{\sqrt{-\Delta}}{|A_n|} \leq \sqrt{\eta^2 - \Delta} + \sqrt{-\Delta}, \\ |z_n| &\geq \frac{1}{\sqrt{-\Delta} + \sqrt{\eta^2 - \Delta}}. \end{aligned}$$

We also have

$$a_{n+1} + z_{n+1} = \frac{1}{z_n}$$

so that

$$|a_{n+1}| \leq |z_{n+1}| + \frac{1}{|z_n|} \leq \rho + \sqrt{-\Delta} + \sqrt{\eta^2 - \Delta}.$$

From Theorem 5.3.1, it follows that $|z - p/q| \geq C'/|q|^2$ for all $p/q \in K$, where

$$C' = \frac{1}{\alpha(\beta + 1)(\beta + \rho + 1)}, \ \beta = \rho + \sqrt{-\Delta} + \sqrt{\eta^2 - \Delta}.$$

$\square$

We conclude this section with an example of the above over $\mathbb{Q}(i)$. The number

$$\sqrt{3}e^{2\pi i/5} = [1+2i; -1+i, -3, 2+2i, -1+3i, -2, -2i, 2+2i, 3-i, -2+2i, \ldots]$$

is a zero of the anisotropic indefinite integral binary Hermitian form $H(z,w) = |z|^2 - 3|w|^2$. Data from 10,000 iterations of the continued fraction algorithm and bounds from Corollary 5.7.4 are:

$$\max_{0 \le n < 10000} \{|a_n|\} = 4.47213\ldots \le 7.22749\ldots,$$

$$\min_{0 \le n < 10000} \{|z_n|\} = 0.25201\ldots \ge 0.15336\ldots,$$

$$\min_{0 \le n < 10000} \{|q_n(q_n z - p_n)|\} = 0.28867\ldots \ge 0.00563\ldots.$$

There are 64 distinct partial quotients a_n and 56 distinct forms H_n. The remainders z_n are shown on the left in Figure 5.2 below. The remainders appear to have different densities along the arcs $Z(H_n) \cap V$, spending more time on the circles of radius $\sqrt{3}$ (show in black in Figure 5.2) than on the circles of radius $\sqrt{3}/2$ (shown in red in Figure 5.2). The bound on the normalized error seems to be rather poor, but it is interesting to note that the minimum $0.28867\ldots$ above agrees with $\frac{1}{2\sqrt{3}}$ to high precision. We give some explanation for this last point in 5.9, where better bounds lower bound for $\liminf_{|q|\to\infty}\{|q(qz-p)|\}$ are given for zeros of Hermitian forms.

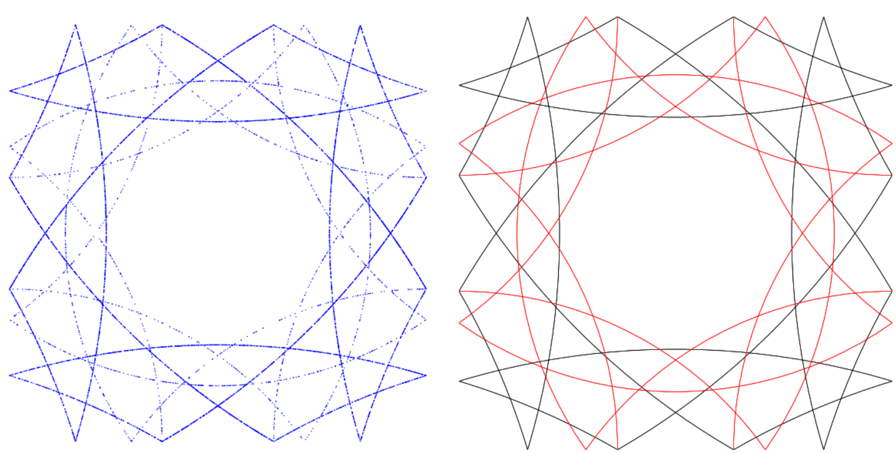

Figure 5.2: The first 10,000 remainders z_n of $z = \sqrt{3}e^{2\pi i/5}$ over $\mathbb{Q}(i)$ (left) and the arcs $Z(H_n) \cap V$ (right).

5.8 Examples via right-angled continued fractions

We describe a "reduction theory" of sorts for binary Hermitian forms over $\mathbb{Q}(i)$. Call H *reduced* if $S(H) \cap \mathcal{F} \neq 0$, where $S(H)$ is the geodesic hemisphere with ideal boundary $Z(H)$ and $\mathcal{F}$ is the fundamental octahedron.

Proposition 5.8.1. *There are only finitely many reduced indefinite integral binary Hermitian forms H of a given determinant $D = \det(H)$.*

Proof. If $a = 0$ and $S(H)$ is a vertical plane, then $D = -|b|^2$ and there are only finitely many choices for b. Also, the line $Z(H) = \mathrm{Re}(z\bar{b}) = c$ must intersect the unit square, so there are only finitely many choices for c. For $a \neq 0$, we will show that there are only finitely many reduced forms intersecting the part of $\mathcal{F}$ near infinity (say $\mathcal{F}_\infty$, the sixth of $\mathcal{F}$ above the unit hemispheres centered at $0, 1, i, 1+i$). By symmetry (the existence of isometries permuting the six cuspidal pieces), this will suffice. The radius of $Z(H)$, $-D/a^2$, is bounded above by $-D$ and bounded below by $1/\sqrt{2}$ since it intersects $\mathcal{F}_\infty$. Hence $|a|$ is bounded. The center of $Z(H)$, b/a is also bounded (else $S(H)$ will not intersect $\mathcal{F}_\infty$). Hence $|b|$ is bounded. Therefore, there are only finitely many reduced forms. $\square$

Now, if z_0 is a zero of a reduced indefinite integral binary Hermitian form H, then applying the transformations associated to $T^n(z_0)$ gives a sequence of Hermitian forms H_n which are also reduced. Hence the orbit $\{H_n\}_{n\geq 0}$ is finite. If H is anisotropic, then the finitely many boundary circles of the finitely many forms stay away from the fixed points, and z_0 will be badly approximable. See Figure 5.3 for an example of one of these special orbits.

A similar argument can be given for the cubeoctahedral case. See Figure 5.4 for an example orbit (under T_B). [Note that 7 is not a norm from $\mathbb{Q}(\sqrt{-1})$ nor $\mathbb{Q}(\sqrt{-2})$ so that the orbits are bounded away from the fixed points in Figures 5.3 and 5.4.]

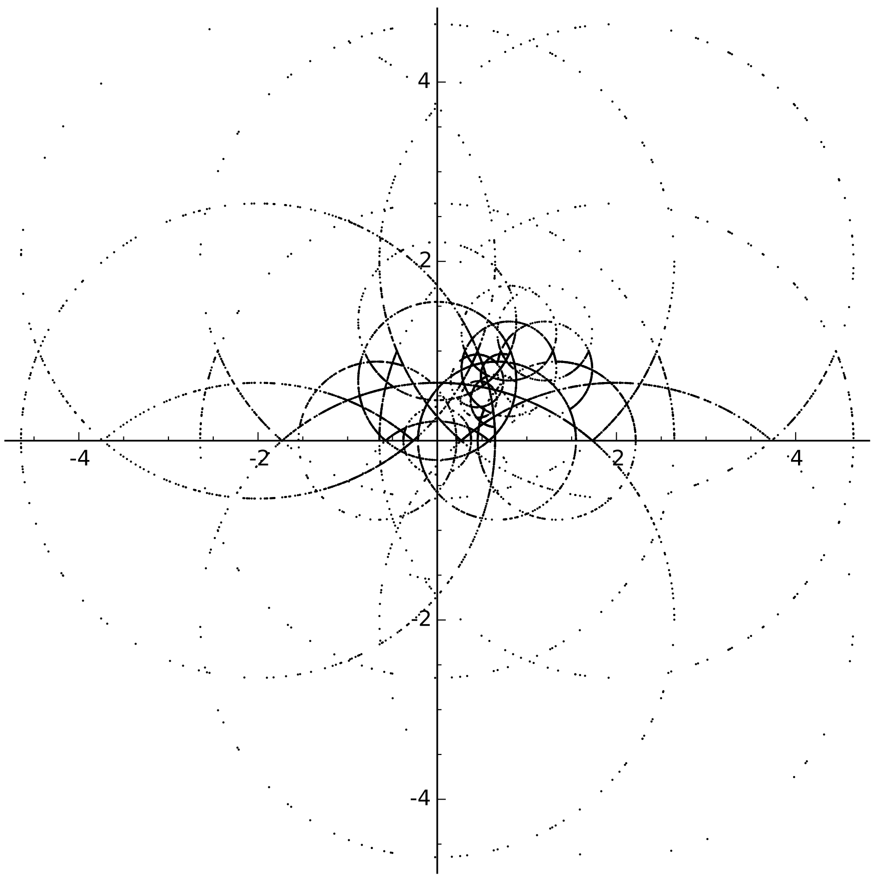

Figure 5.3: 10000 iterations of $T = T_B$ over $\mathbb{Q}(\sqrt{-1})$ on $z = \sqrt{7}(\cos(2\pi/5) + i\sin(2\pi/5))$.

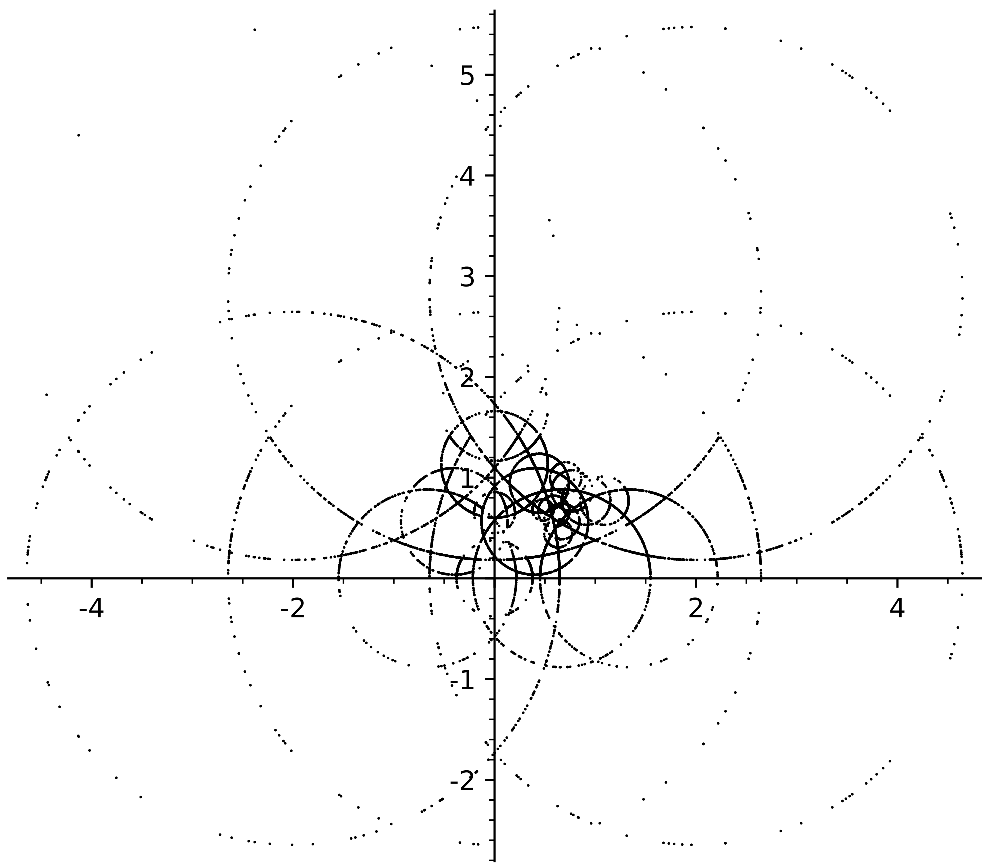

Figure 5.4: 10000 iterations of T_B over $\mathbb{Q}(\sqrt{-2})$ on $z = \sqrt{7}(\cos(2\pi/5) + i\sin(2\pi/5))$.

5.9 Examples à la Liouville

Finally, we note that Proposition 5.6.1 and 5.6.3 have elementary proofs along the lines of Liouville's theorem. Let J be a totally indefinite, anisotropic, integral binary quadratic or Hermitian form. For $\mathbf{z} \in Z(J)$ and $p/q \in F$ with $\max_i\{|z_i - p_i/q_i|\} \leq 1$, we have

$$|J_i(p_i/q_i, 1)| = |J(z_i, 1) - J(p_i/q_i, 1)| \leq \kappa_i |z_i - p_i/q_i|$$

for some constant $\kappa_i > 0$ depending on z_i and J_i by the mean value theorem. Multiplying by $|q_i|^2$ we have

$$|J_i(p_i, q_i)| \leq \kappa_i |q_i(q_i z_i - p_i)|.$$

Because J is anisotropic and integral, for any $0 < \lambda \leq \min\{\max_i\{|a_i|\} : 0 \neq a \in \mathcal{O}_F\}$ we have

$$\max_i\{|J_i(p_i, q_i)|\} \geq \lambda.$$

Hence for some i_0 we have

$$\lambda \leq \kappa_{i_0} |q_{i_0}(q_{i_0} z_{i_0} - p_{i_0})| \leq \kappa_{i_0} \max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\},$$

and $\mathbf{z}$ is badly approximable

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \geq \lambda \kappa_{i_0}^{-1}.$$

We refine the above argument for Hermitian forms in the imaginary quadratic case. For the related topic of approximation properties of quadratic irrationals in $Z(H)$, see [Vul10], Theorem 1.1.

Theorem 5.9.1. *Let $H(z, w) = Az\bar{z} - B\bar{z}w - \overline{B}z\bar{w} + Cw\bar{w}$ be a binary Hermitian form that is indefinite, anisotropic, and integral. Let $\Delta(H) = AC - |B|^2 < 0$ be the determinant of H and*

$$\mu(H) = \min\{|H(p, q)| : (0, 0) \neq (p, q) \in \mathcal{O}_K^2\} > 0$$

its absolute minimum. If $H(z, 1) = 0$, then z is badly approximable with

$$|q(qz - p)| \geq \frac{\mu(H)}{2\sqrt{-\Delta} + 2|A|\epsilon}$$

for any $p/q \in K$ with $|z - p/q| \le \epsilon$. Hence

$$\liminf_{|q|\to\infty}\{|q(qz-p)| : p, q \in \mathcal{O}_K, q \neq 0\} \ge \frac{\mu(H)}{2\sqrt{-\Delta(H)}}.$$

Proof. We will apply the mean value theorem to an $f : \mathbb{R}^2 \to \mathbb{R}$ in the form

$$f(b_1, b_2) - f(a_1, a_2) = \left(\frac{\partial f}{\partial x}(c_1, c_2), \frac{\partial f}{\partial y}(c_1, c_2)\right) \cdot (b_1 - a_1, b_2 - a_2)$$

for some (c_1, c_2) on the line segment joining (a_1, a_2) and (b_1, b_2). By the mean value theorem, the Cauchy-Schwarz inequality, and because $H(z, 1) = 0$, we have

$$\begin{aligned}|H(p/q, 1)| &= |H(p/q, 1) - H(z, 1)| \le |2A(c_1 + c_2 i) - 2B||z - p/q| \\ &= 2|A||(c_1 + c_2 i) - B/A||z - p/q| \le (\sqrt{-\Delta} + |A|\epsilon)|z - p/q| \\ &= C_\epsilon |z - p/q|,\ C_\epsilon = 2\sqrt{-\Delta(H)} + 2|A|\epsilon,\end{aligned}$$

for any $p/q \in K$ with $|z - p/q| \le \epsilon$. (Above we are using the fact that the point $c_1 + c_2 i$ is at most ϵ from the disk of radius $\sqrt{-\Delta}/|A|$ centered at B/A.) Multiplying by $|q|^2$ gives

$$0 < \mu(H) \le |A|p|^2 - B\bar{p}q - \overline{B}p\bar{q} + C|q|^2| \le C_\epsilon |q|^2 |z - p/q|,$$

since H is anisotropic and integral. Therefore z is badly approximable with

$$|q(qz - p)| \ge \mu(H)/C_\epsilon > 0.$$

Letting ϵ tend to zero gives

$$\liminf_{|q|\to\infty}\{|q(qz-p)| : p, q \in \mathcal{O}_K, q \neq 0\} \ge \frac{\mu(H)}{2\sqrt{-\Delta(H)}}.$$

□

5.10 Non-quadratic algebraic examples over CM fields

It should be emphasized that the badly approximable product of circles $Z(H) \subseteq \mathbb{C}^n$ contains non-quadratic algebraic vectors, parameterized as follows (cf. [Hin18b]).

Corollary 5.10.1. *Choose real algebraic numbers* $\alpha_i \in [-2, 2]$, $1 \le i \le n$, $f \in F$, *and a totally positive* $e \in E \setminus N_E^F(F)$. *Then the vectors*

$$\mathbf{z} = (z_1, \ldots, z_n), \;\; z_i = f_i + \sqrt{e_i} \cdot \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4}}{2}$$

are algebraic and badly approximable, where $f_i = \sigma_i(f)$, $e_i = \sigma_i(e)$.

We discuss the above over imaginary quadratic fields K; the general case follows the same argument. For z such that $|z|^2 = s/t \in \mathbb{Q}$ is not a norm from K, the anisotropic integral form $\begin{pmatrix} t & 0 \\ 0 & -s \end{pmatrix}$ shows that z is badly approximable. This essentially exhausts all of the examples we've given as we can translate an integral form by a rational to center it at zero, then clear denominators to obtain an integral form as described, i.e.

$$H = \begin{pmatrix} A & -B \\ -\overline{B} & C \end{pmatrix}, \; g = \begin{pmatrix} 1 & -B/A \\ 0 & 1 \end{pmatrix}, \; A \cdot {}^g H = \begin{pmatrix} A^2 & 0 \\ 0 & AC - B\overline{B} \end{pmatrix}.$$

For some specific algebraic examples, consider quadratically scaled roots of unity $z = \sqrt{n}\zeta$ for $|z|^2 = n \in \mathbb{Q}$ not a norm, or the generalizations of examples from [BG12], $z = \sqrt[m]{a} + \sqrt{\sqrt[m]{a^2} - n}$ for $a \in \mathbb{Q}$, $\sqrt[m]{a^2} < n$, and $n = |z|^2$ not a norm. See Figures 5.5, 5.6 for visualizations of the orbits of algebraic numbers satisfying $|z|^2 = n$ for various n and $d = 1, 3$.

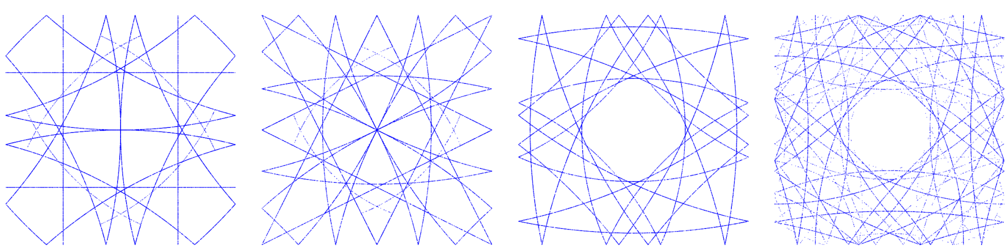

Figure 5.5: 20,000 iterates of T for $z = \sqrt[3]{2} + \sqrt{\sqrt[3]{4} - n}$ over $\mathbb{Q}(\sqrt{-1})$ with $n = 4, 5, 6, 7$.

We would like to characterize the badly approximable algebraic numbers captured above. One such characterization comes from a parameterization of the algebraic numbers on the unit circle (taken from the mathoverflow post [Con10]).

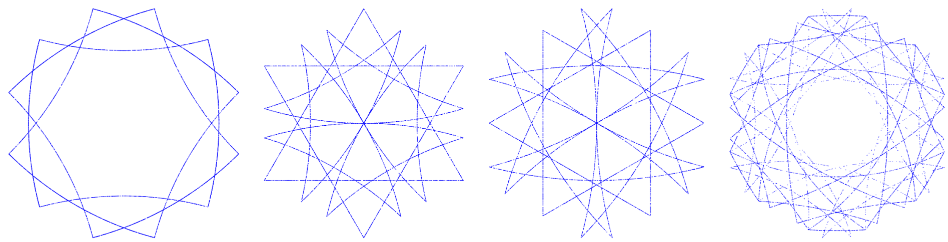

Figure 5.6: 10,000 iterates of T for $z = \sqrt[3]{2} + \sqrt{\sqrt[3]{4} - n}$ over $\mathbb{Q}(\sqrt{-3})$ with $n = 2, 3, 4, 5$.

Lemma 5.10.2 ([Con10])**.** *The algebraic numbers w on the unit circle, $w\overline{w} = 1$, are those numbers of the form*

$$w = \frac{u \pm \sqrt{u^2 - 4}}{2}$$

where u is a real algebraic number in the interval $[-2, 2]$. If $u \neq \pm 2$, the minimal polynomial f of w is

$$f(t) = t^m g(t + 1/t)$$

where $g(t)$ is the minimal polynomial of u, $\deg(g) = m$*. In particular, the degree of f is even.*

Proof. We know that $w, \overline{w} = 1/w$ have the same minimal polynomial $f(x) \in \mathbb{Q}[x]$, say of degree d. One can deduce that $x^d f(1/x) = f(x)$ so that $f(x)$ is palindromic (if $f(x) = \sum_k f_k x^k$ then $f_{d-k} = f_k$) and since the roots come in reciprocal pairs, d is even. The even degree palindromic polynomials are of the form $f(x) = x^{d/2} g(x + 1/x)$ for some polynomial g

$$\begin{aligned} f(x) = \sum_{k=0}^{d} f_k x^k &= x^{d/2} \sum_{k=0}^{d/2} f_k (x^k + 1/x^k) \\ &= x^{d/2} \left(r(x + 1/x) + \sum_{k=0}^{d/2} f_k (x + 1/x)^k \right) \\ &= x^{d/2} g(x + 1/x),\ g(x) = r(x) + \sum_{k=0}^{d/2} f_k x^k, \end{aligned}$$

noting that the difference $(x + 1/x)^k - (x^k + 1/x^k)$ is a polynomial in $x + 1/x$ by symmetry of the binomial coefficients. The roots of the even degree palindromic f on the circle double cover the roots of g in the interval $(-2, 2)$ (via $w = e^{i\theta} \mapsto u = 2\cos\theta$). Conversely, taking an irreducible polynomial $g(x) \in \mathbb{Q}[x]$ of degree $d/2$ with a root in the interval $(-2, 2)$ gives a degree d irreducible polynomial $f(x) = x^{d/2} g(x + 1/x)$ with roots on the unit circle. $\square$

Hence we have the following corollary describing the badly approximable algebraic numbers coming from (indefinite anisotropic K-rational binary) Hermitian forms over imaginary quadratic fields.

Corollary 5.10.3. *For any imaginary quadratic field K, there are algebraic numbers of each even degree over $\mathbb{Q}$ that are badly approximable over K. Specifically, for any real algebraic number $u \in [-2, 2]$, any positive $n \in \mathbb{Q} \setminus N_{\mathbb{Q}}^{K}(K)$, and any $t \in K$, the number*

$$z = t + \sqrt{n} \cdot \frac{u \pm \sqrt{u^2 - 4}}{2}$$

is badly approximable.

For instance, the examples in Figures 5.5 and 5.6 have $t = 0$ and $u = 2^{4/3}/\sqrt{n}$.

Chapter 6

Continued fractions for weakly Euclidean orders in definite Clifford algebras

6.1 Introduction

The "accidental" isomorphisms

$$\mathrm{Isom}^+(\mathcal{H}^2) = O_{\mathbb{R}}(1,2)^\circ \cong PSL_2(\mathbb{R}),\ \mathrm{Isom}^+(\mathcal{H}^3) = O_{\mathbb{R}}(1,3)^\circ \cong PSL_2(\mathbb{C})$$

relate two- and three-dimensional hyperbolic geometry and simultaneous approximation over a number field F via the arithmetic lattice $SL_2(\mathcal{O}_F)$.

Can one use hyperbolic geometry in higher dimensions to approximate $x \in \mathbb{R}^n \subseteq \partial\mathcal{H}^{n+1}$ in a similar fashion? One can describe $\mathcal{H}^{n+1}$ as a subset of a Clifford algebra and $\mathrm{Isom}^+(\mathcal{H}^{n+1})$ as a group of two-by-two matrices (with entries in the Clifford algebra) acting as fractional linear transformations on this subset and as projective tranformations on S^n as a sort of projective line over this Clifford algebra. This generalizes the two cases above, where the Clifford algebras are $\mathbb{C}$ and $\mathbb{H}$ in dimensions 2 and 3 respectively.

Given an order $\mathcal{O}$ in this Clifford algebra, we can approximate $x \in \mathbb{R}^n$ as a quotient of two "integers" $p, q \in \mathcal{O}$ such that $pq^{-1} \in \mathbb{R}^n$. This too generalizes the above, although it seems less natural, using integers in a 2^n-dimensional algebra to approximate vectors in $\mathbb{R}^n$.

In 6.6, we look in particular at the order $\mathbb{Z}[i, j, ij] \subseteq \mathbb{H}$ (i.e. approximation in $\mathbb{R}^3$), giving a convergent nearest integer continued fraction algorithm. We prove (Theorem 6.6.1) that the partial quotients of zeros of (anisotropic indefinite rational binary) Hermitian forms are bounded under the assumption that the convergent denominators are increasing for the algorithm (in the same manner as Theorem 5.7.2). We also give a Liouville type argument that such zeros of Hermitian forms are badly approximable in Theorem 6.6.2 (in the same manner as Theorem 5.9.1).

References for the "SL_2" model of hyperbolic isometries include [Wat93], [Ahl84], [Ahl85], [Ahl86] [Por95], [McI16], [Lou01]. Some work on continued fractions and approximation in this setting have been considered as well: [Bea03], [LV18], [Sch69], [Sch74], [Vul93], [Vul95a], [Vul99]. More discussion and application of arithmetic subgroups in this "SL_2" isometry group can be found in: [EGM87], [EGM88], [EGM90], [Maa49], [MWW89].

6.2 Clifford algebras

Let V be a finite dimensional vector space over a field K (of characteristic not equal to 2) and $Q : V \to K$ a quadratic form ($Q(\alpha v) = \alpha^2 Q(v)$), with associated symmetric blinear form $2B(v, w) = Q(v + w) - Q(v) - Q(w)$. We will furthermore assume throughout that B is non-degenerate ($B(w, V) = 0 \Rightarrow w = 0$). The *Clifford algebra* $\text{Cliff}(V, Q)$ associated to Q is $T(V)/(v^2 = Q(v))$, the tensor algebra over V modulo the ideal generated by forcing squaring $v \in V$ to be evaluation of Q at v. Let $e_1, \ldots, e_n$ be an orthogonal basis for the quadratic space (V, Q). Then $\text{Cliff}(V, Q)$ has dimension 2^n with basis $\{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$ and relations generated by $e_i^2 = Q(e_i)$, $e_i e_j = -e_j e_i$.

We have three involutions x', x^*, and $\bar{x} = (x')^* = (x^*)'$ whose action on the basis elements are

$$(e_{i_1} \cdots e_{i_k})' = (-e_{i_1}) \cdots (-e_{i_k}) = (-1)^k e_{i_1} \cdots e_{i_k},$$
$$(e_{i_1} \cdots e_{i_k})^* = e_{i_k} \cdots e_{i_1} = (-1)^{\frac{k(k-1)}{2}} e_{i_1} \cdots e_{i_k},$$
$$\overline{e_{i_1} \cdots e_{i_k}} = (-1)^{\frac{k(k+1)}{2}} e_{i_1} \cdots e_{i_k},$$

with * and $^-$ reversing the order of multiplication.

Note that the anisotropic vectors $Q(v) \neq 0$ are invertible, $v^2 = Q(v) \neq 0$ implies $v^{-1} = v/Q(v)$, and that $v\bar{v} = vv' = -Q(v)$. This extends to products of vectors, $x\bar{x} = (-1)^k Q(v_1) \cdots Q(v_k)$ if $x = v_1 \cdots v_k$. Hence the collection of products of anisotropic vectors forms a group, $G(V, Q)$.

This is a (rather small) subgroup of $\mathrm{Cliff}(V,Q)^\times$, but it is directly related to the orthogonal group $O(V,Q)$ as follows.

Proposition 6.2.1. *For $a \in \mathrm{Cliff}(V,Q)^\times$ and $v \in V$, let $\rho_a(v) = a'va^{-1}$. Then $\rho_a \in GL(V)$ iff $a \in G(V,Q)$. Moreover, for $w \in V$, $\rho_w(v) = v - 2\frac{B(v,w)}{Q(w)}w$ is the reflection with respect to v. Hence we have an isomorphism*

$$\rho : G(V,Q)/K^\times \to O(V,Q),\ aK^\times \mapsto \rho_a.$$

6.3 A special case

We will be working with $K = \mathbb{R}$, $V = \mathbb{R}^{n-1}$, $Q(x) = -\sum_{k=1}^{n-1} x_i^2$, with $\mathrm{Cliff}(V,Q) =: \mathcal{C}_n$. We will identify $\mathbb{R}^n$ with $\langle 1, e_1, \ldots, e_{n-1}\rangle_{\mathbb{R}} \subseteq \mathcal{C}_n$. In this case, we have $O(V,Q) \cong G_n^1/\pm 1$, where $G_n^1 = \{x \in G_n : x\bar{x} = 1\}$. We also note that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ in an obvious way. We extend Q to $\mathbb{R}^n$ by $\widetilde{Q} = x_0^2 - Q(x_1,\ldots,x_{n-1}) = \sum_{i=0}^{n-1} x_i^2 = x\bar{x} = |x|^2$. Let $\widetilde{G}_n$ be the group generated by $\mathbb{R}^n$. We then have the following.

Proposition 6.3.1. *There is an isomomorphism*

$$\widetilde{\rho} : \widetilde{G}_n/\mathbb{R}^\times \to SO(\mathbb{R}^n, \widetilde{Q}),\ a\mathbb{R}^\times \mapsto axa'^{-1}.$$

A reflection σ_y perpendicular to y is given by $x \mapsto -y\bar{x}\bar{y}^{-1} = -yx'y'^{-1}$, and the composition of two reflections $\sigma_{y_2} \circ \sigma_{y_1}$ perpendicular to $y_1, y_2 \in \mathbb{R}^n$ is given by ρ_a where $a = y_2y_1^{-1}$ or y_2y_1'.

Proof. We have

$$\begin{aligned}\sigma_y x &= x - 2\widetilde{B}(x,y)\frac{y}{\widetilde{Q}(y)} \\ &= x - (x\bar{y} + y\bar{x})\bar{y}^{-1} \\ &= -y\bar{x}\bar{y}^{-1}.\end{aligned}$$

Composing two of these gives ρ_a as stated, noting that $^-$ and $'$ are equal on $\mathbb{R}^n$, and $SO(\mathbb{R}^n, \widetilde{Q})$ is the collection of all even products of reflections. Those $a \in \widetilde{G}_n$ such that $axa'^{-1} = x$ for all $x \in \mathbb{R}^n$ are the non-zero reals; we have $a = a'$ so a is even, and $ax = xa$ with $x = e_i$ shows that a cannot contain monomials with e_i. $\square$

6.4 S^n as a projective line and the Vahlen group

Consider the set of pairs $(x, y) \in \widetilde{G}_n$ such that $y = 0$ or $xy^{-1} \in \mathbb{R}^n$, and call two such pairs equivalent if there exists $z \in \widetilde{G}_n$ such that $(x, y) = (xz, yz)$ (scalars on the right). We can identify these equivalence classes with $\mathbb{R}^n \cup \{\infty\}$ via $[x : y] \sim [xy^{-1} : 1]$ or $[x : 0] \sim [1 : 0] =: \infty$, or with S^n, the unit n-sphere. We want to identify a subgroup of $GL_2(\mathcal{C}_n)$ that acts as fractional linear transformations on $\mathbb{R}^n \cup \{\infty\}$ or projective automorphisms (matrix multiplication on the left)

$$[x : y] \mapsto [ax + by : cx + dy],\ xy^{-1} \mapsto (axy^{-1} + b)(cxy^{-1} + d)^{-1}.$$

If the matrix g induces the identity function, then it is diagonal with entries in $Z(\mathcal{C}_n) \setminus \{0\}$, where the center $Z(\mathcal{C}_n)$ is $\mathbb{R}$ if n is odd and $\mathbb{R} + \mathbb{R}e_1 \cdots e_{n-1}$ if n is even. One can check that composition of functions and matrix multiplication agree. If $y = (ax + b)(cx + d)^{-1}$ is bijective with $x, y \in \mathbb{R}^n$, then $x = (-d^*y + b^*)(c^*y - a^*)^{-1}$ and composing in either order gives equality of functions

$$1 = \begin{pmatrix} -ad^* + bc^* & ab^* - ba^* \\ -cd^* + dc^* & cb^* - da^* \end{pmatrix} = \begin{pmatrix} -d^*a + b^*c & -d^*b + b^*d \\ c^*a - a^*c & c^*b - a^*d \end{pmatrix}.$$

Therefore we have

$$ab^* = ba^*,\ cd^* = dc^*,\ a^*c = c^*a,\ b^*d = d^*b$$

$$\Delta = ad^* - bc^* = da^* - cb^* \in Z(\mathcal{C}_n),\ \Delta' = d^*a - b^*c = a^*d - c^*b \in Z(\mathcal{C}_n).$$

If g is bijective, then the conditions above show that Δ, Δ' are multiplicative. Also $g(0) = bd^{-1}$, $g(\infty) = ac^{-1}$, $g^{-1}(0) = -b^*(a^*)^{-1}$, $g^{-1}(\infty) = -d^*(c^*)^{-1}$ are all in $\mathbb{R}^n$.

Now if g_1, g_2 have entries in $\widetilde{G}_n \cup \{0\}$, then the product also has entries in $\widetilde{G}_n \cup \{0\}$, e.g. $a_1a_2 + b_1c_2 = b_1(b_1^{-1}a_1 + c_2a_2^{-1})a_2$. One can verify that if $a, c \in \widetilde{G}_n \cup \{0\}$, then $ac^{-1} \in \mathbb{R}^n$ iff $ac^* \in \mathbb{R}^n$.

Some convenient generators for the (orientation preserving) Möbius transformations are special orthogonal transformations, translations, similarities, and $-1/x$. In matrices these are

$$\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix},\ a \in \widetilde{G}_n,\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ x \in \mathbb{R}^n,\ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix},\ t \in \mathbb{R},\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 6.4.1. *The collection* $SV_n(\mathbb{R})$ *of matrices*[1] $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that*

$$a, b, c, d \in \widetilde{G}_n \cup \{0\},\ \Delta(g) = ad^* - bc^* = 1,\ ac^*, bd^* \in \mathbb{R}^n\ (ac^{-1}, bd^{-1} \in S^n)$$

form a group, acting bijectively on $\mathbb{R}^n \cup \{\infty\}$ *via* $(ax+b)(cx+d)^{-1}$. *The fractional linear action extends to* $\mathcal{H}^{n+1} = \{x_0 + x_1e_1 + \ldots + x_ne_n\ :\ x_i \in \mathbb{R},\ x_n > 0\} \subseteq \mathcal{C}_{n+1}$ *and* $PSV_n(\mathbb{R}) = SV_n(\mathbb{R})/\{\pm 1\}$ *is isomorphic to* $\mathrm{Isom}^+(\mathcal{H}^{n+1})$.

Proof. If $c \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (c^*)^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

For more details see [Wat93]. □

6.5 Continued fractions for weakly Euclidean orders

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice and suppose there is a fundamental domain $\mathcal{F} \subset \mathbb{R}^n$ for the action of $(\Lambda, +)$ on $\mathbb{R}^n$ such that $\mathcal{F} \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$. Then we have a "Euclidean algorithm" for pairs $a, b \in \Lambda$, $ab^{-1} \in \mathbb{R}^n$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix},$$

where $a_0 \in \Lambda$ is the lattice point such that $x = ab^{-1} = a_0 + x_0$, $x_0 \in \mathcal{F}$, a_1 is the lattice point such that $(x - a_0)^{-1} = a_1 + x_1$, $x_1 \in \mathcal{F}$, etc. The element r is a "right GCD" for a, b in that

$$a = p_nr,\ b = q_nr,\ r = q_{n-1}^*a - p_{n-1}^*b.$$

Define $T : \mathcal{F} \to \mathcal{F}$ by $Tx = y$ where $1/x = a + y$, $a \in \Lambda$, $y \in \mathcal{F}$. For $x \in \mathbb{R}^n$ let $x = a_0 + x_0$, $x_n = T^nx_0$, $a_n + x_n = 1/x_{n-1}$. Then

$$xq_n - p_n = (-1)^nx_0x_1 \cdots x_n.$$

[1] The Vahlen group, after K. Th. Vahlen [Vah02]. He was an unrepentant Nazi, perhaps the reason why his name is sometimes avoided in the literature.

We will furthermore assume that $\Lambda = \mathcal{O} \cap \mathbb{R}^n$ where $\mathcal{O}$ is a discrete subring of $\mathcal{C}_n$ closed under *, so that the matrices coming from iteration of T and their inverses have entries in $\mathcal{O}$. If the order $\mathcal{O}$ is Euclidean, then the conditions above are statisfied, but it isn't necessary. We will say that such orders satisfy the *weak Euclidean property.*

If $\mathcal{O}$ satisfies the weak Euclidean property, then we have the following.

Proposition 6.5.1. *For $x \in \mathbb{R}^n \setminus \{pq^{-1} : p, q \in \mathcal{O}\}$, $p_n q_n^{-1}$ converges to x (exponentially in n). If $x = ab^{-1} \in \mathbb{R}^n \cap \mathcal{O}\mathcal{O}^{-1}$, then for some n, $x_n = 0$ and $p_n/q_n = x$. If the denominators q_n are strictly increasing in norm, then there exists $C > 0$ such that $|x - p_n q_n^{-1}| < C/|q_n|^2$ for all x and n.*

Proof. Since $q_n \in \mathcal{O}$ is discrete, q_n is uniformly bounded below unless $q_n = 0$. If n is least such that $q_n = 0$, then $x_n = 0$ and $x \in \mathcal{O}\mathcal{O}^{-1}$. For $x \in \mathbb{R}^n \setminus \mathcal{O}\mathcal{O}^{-1}$, the equality $xq_n - p_n = (-1)^n x_0 x_1 \cdots x_n$ then shows that $|x - p_n q_n^{-1}| \leq mM^n$ where $M = \sup_{v \in \mathcal{F}}\{|v|\} < 1$ and $m = \min_{a \in \mathcal{O}}\{|a|\}$.

For $x = ab^{-1} \in \mathbb{R}^n \cap \mathcal{O}\mathcal{O}^{-1}$, the weak Euclidean property gives a "Euclidean algorithm"

$$a = q_1 b + r_1, \;\; q_1, r_1 \in \mathcal{O}, \;\; |r_1| < |b|$$

$$b = q_2 r_1 + r_2, \;\; q_2, r_2 \in \mathcal{O}, \;\; |r_2| < |r_1|$$

etc.,

terminating in finite time since $\mathcal{O}$ is discrete. This shows, taking the quotient $q = a_n$ and remainder $r_n = x_n$ at each step to be as given by the continued fraction algorithm, that the continued fraction algorithm terminates for $x = ab^{-1} \in \mathcal{O}\mathcal{O}^{-1}$.

Finally, taking norms of the equality

$$x - p_n q_n^{-1} = (-1)^n x_0 x_1 \cdots x_n = (-1)^n (q_n^*)^{-1} (x_n^{-1} + q_n^{-1} q_{n-1})^{-1} q_n^{-1}$$

and bounding $|x_n^{-1}| > M > 1$, $|q_n^{-1} q_{n-1}| < m < 1$ gives

$$|x - p_n q_n^{-1}| \geq \frac{1}{(M - m)|q_n|^2}.$$

$\square$

6.6 Approximation in $\mathbb{R}^3$ via quaternions

Let $\mathcal{O} = \mathbb{Z}[i, j, ij]$ be the Lipschitz quaternions. Then $\Lambda = \mathcal{O} \cap \mathbb{R}^3 = \mathbb{Z}^3$ has the weak Euclidean property (the unit cube centered at zero is contained in the unit sphere). For $x \in \mathbb{R}^3 \setminus \{ab^{-1} : a, b \in \mathcal{O}\}$, we obtain a convergent infinite continued fraction and rational approximations $p_n q_n^{-1} \in \mathcal{O}\mathcal{O}^{-1} \cap \mathbb{R}^3 \subseteq \mathbb{Q}^3$

$$x = [a_0; a_1, a_2, \ldots],\ p_n q_n^{-1} = [a_0; a_1, \ldots, a_n],\ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$x = (p_n + p_{n-1}x_n)(q_n + q_{n-1}x_n)^{-1},\ xq_n - p_n = (-1)^n x_0 x_1 \cdots x_n = (-1)^n (q_n^*)^{-1}(x_n^{-1} + q_n^{-1}q_{n-1})^{-1}.$$

Assuming the denominators are increasing in norm, $|q_n| > |q_{n-1}|$, all of the convergents satisfy $|x - p_n q_n^{-1}| < C/|q_n|^2$ for some constant $C > 0$. This seems to be true experimentally (cf. Figure 6.1) and a proof analogous to those in 3.3.1 could probably be given with some effort. We also have $SV(\mathcal{O}) \cdot \infty = \mathbb{Q}^3$ in this case (i.e. every rational triple can be expressed as ab^{-1} for some $a, b \in \mathcal{O}$).

6.6.1 Badly approximable triples from Hermitian forms

Let $A, C \in \mathbb{R}$ and $B \in \mathbb{R}^n \subseteq \mathcal{C}_n$ with $\Delta = AC - |B|^2 < 0$. For $[x : y] \in S^n$ we consider the zero sets $Z(H)$ of indefinite Hermitian forms

$$H(x, y) = (\bar{x}, \bar{y}) \begin{pmatrix} A & B \\ \bar{B} & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which are spheres $S^{n-1} \subseteq S^n$. If $A \neq 0$, then the zero set is given by

$$\{x : H(x, 1) = 0\} = \{x \in \mathbb{R}^n : |x + B/A|^2 = -\Delta/A\}.$$

The group SV_n has a right action on such H, $H^g = H \circ g = g^\dagger H g$, where $g^\dagger$ is the conjugate transpose, ($\bar{}$ each entry and transpose the matrix). We have equality of zero sets $Z(H^g) =$

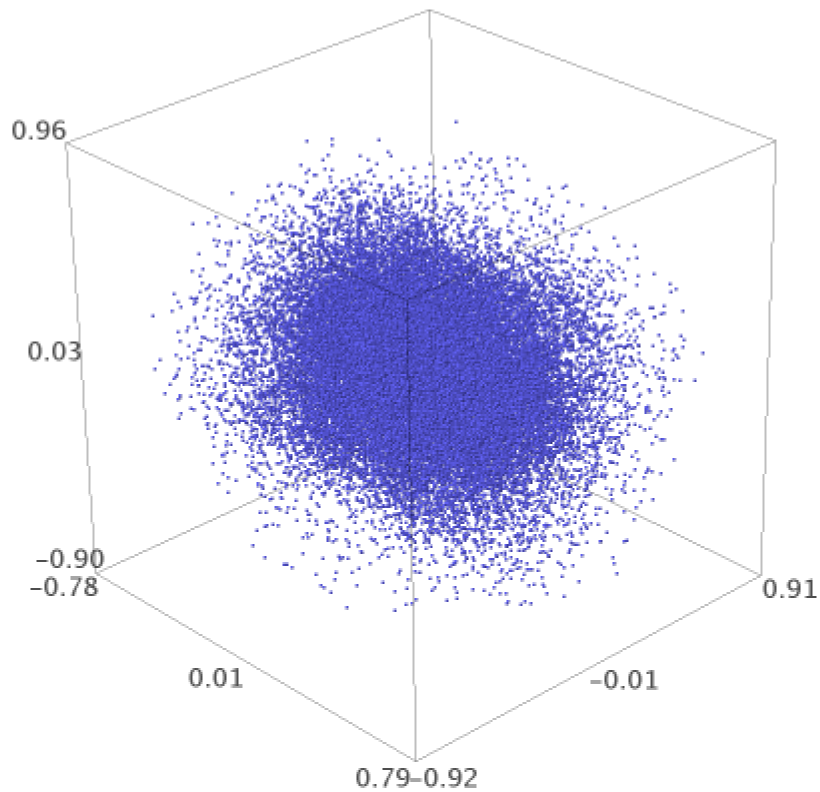

Figure 6.1: The ratios $q_{n-1}q_n^{-1}$, $1 \leq n \leq 10$, for 5000 randomly chosen points of $[-1/2, 1/2)^3$. All of the points lie in the unit sphere, giving experimental evidence that the convergent denominators are increasing.

$g^{-1}Z(H)$. The quadratic form $\Delta(H) = AC - |B|^2$ is signature $(1, n)$ and is SV_n-invariant. For reference, we have

$$H^g = \begin{pmatrix} A|\alpha|^2 + \bar{\gamma}\bar{b}\alpha + \bar{\alpha}B\gamma + \bar{\gamma}C\gamma & A\bar{\alpha}\beta + \bar{\gamma}\overline{B}\beta + \bar{\alpha}B\delta + C\bar{\gamma}\delta \\ A\bar{\beta}\alpha + \bar{\delta}\overline{B}\alpha + \bar{\beta}B\gamma + C\bar{\delta}\gamma & A|\beta|^2 + \bar{\delta}\overline{B}\beta + \bar{\beta}B\delta + C|\delta|^2 \end{pmatrix}, \; g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We restrict our attention and consider three-dimensional continued fractions over the Lipschitz quaternions, i.e. $\mathbb{R}^3 \cong \langle 1, i, j \rangle_{\mathbb{R}}$. If $x \in \mathbb{R}^3$ lies on a rational sphere (i.e. $[x : 1]$ is a root of a rational indefinite binary Hermitian form H) and the denominators are increasing for the algorithm, then the orbit $T^n(x)$ is constrained to lie on finitely many spheres (i.e. the zero sets of the orbit of H). If H is anisotropic, then $T^n(x)$ is bounded away from zero. We summarize this as follows.

Theorem 6.6.1. *Assume the convergent denominators are increasing in norm. If x is a zero of an indefinite integral binary Hermitian form H and $H_n = H^{g_n}$ is the form obtained by the change of variable*

$$g_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

then the orbit $\{H_n\}_{n\geq 0}$ is finite. In particular, if $x \in \mathbb{R}^3$ is a zero of an indefinite anisotropic integral binary Hermitian form,

$$H(x, 1) = 0, \;\; H(x, y) = (\bar{x}, \bar{y}) \begin{pmatrix} A & B \\ \bar{B} & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$A, C \in \mathbb{Z}, \;\; B \in \langle 1, i, j \rangle_{\mathbb{Z}}, \;\; \Delta := AC - |B|^2 < 0, -\Delta \textit{ not a sum of three squares,}$$

then the partial quotients of x are bounded. [Recall that n is a sum of three squares unless $n = 4^a(8b + 7)$ for some a, b.]

Proof. Let

$$H_n = H^{g_n} = \begin{pmatrix} A_n & B_n \\ \overline{B}_n & C_n \end{pmatrix}$$

so that

$$\begin{aligned}
A_n &= A|p_n|^2 + \bar{q}_n \overline{B} p_n + \bar{p}_n B q_n + C|q_n|^2, \\
B_n &= A\bar{p}_n p_{n-1} + \bar{q}_n \overline{B} p_{n-1} + \bar{p}_n B q_{n-1} + c\bar{q}_n q_{n-1}, \\
C_n &= A|p_{n-1}|^2 + \bar{q}_{n-1} \overline{B} p_{n-1} + \bar{p}_{n-1} B q_{n-1} + C|q_{n-1}|^2 \\
&= A_{n-1}.
\end{aligned}$$

As noted earlier, we have

$$xq_n - p_n = (-1)^n (q_n^*)^{-1} (x_n^{-1} + q_n^{-1} q_{n-1})^{-1}.$$

Assuming the convergent denominators increase in norm, this gives

$$|xq_n - p_n| \leq \frac{K}{|q_n|}$$

for some $K > 0$. Hence $p_n = xq_n + \kappa q_n^{-1}$ for some $\kappa \in \mathbb{H}$ with $|\kappa| \leq K$. Putting this into the formulas for the coefficients of H_n above, we get

$$\begin{aligned}
A_n = H_n(p_n, q_n) &= H_n((xq_n + \kappa q_n^{-1}, q_n) \\
&= A|xq_n + \kappa q_n^{-1}|^2 + \bar{q}_n \overline{B}(xq_n + \kappa q_n^{-1}) + (\bar{q}_n \bar{x} + \bar{q}_n^{-1} \bar{\kappa}) B q_n + C|q_n|^2 \\
&= \bar{q}_n H(x,1) q_n + A(xq_n \bar{q}_n^{-1} \bar{\kappa} + \kappa q_n^{-1} \bar{q}_n \bar{x} + |\kappa|^2/|q_n|^2) + \bar{q}_n \overline{B} \kappa q_n^{-1} + \bar{q}_n^{-1} \bar{\kappa} B q_n \\
&= 0 + A(xq_n \bar{q}_n^{-1} \bar{\kappa} + \kappa q_n^{-1} \bar{q}_n \bar{x} + |\kappa|^2/|q_n|^2) + \bar{q}_n \overline{B} \kappa q_n^{-1} + \bar{q}_n^{-1} \bar{\kappa} B q_n, \\
|A_n| \leq 2|A||x||\kappa| + |A||\kappa|^2/|q_n|^2 &+ 2|B||\kappa| \leq |A|K^2 + 2|A|K + 2|B|K.
\end{aligned}$$

Therefore the coefficients A_n are bounded. Since $C_n = A_{n-1}$, the coefficients C_n are also bounded. Finally, the action of SV_3 preserves the determinant Δ of the form H, so that $|B_n| = \sqrt{A_n C_n - \Delta}$ is bounded as well. □

We can unconditionally show that zeros of anisotropic indefinite integral binary Hermitian forms are *badly approximable* in the sense that there exists $\lambda > 0$ such that

$$|\zeta - pq^{-1}| \geq \frac{\lambda}{|q|^2}, \ p, q \in \mathbb{Z}[i,j,k], \ pq^{-1} \in \mathbb{R}^3,$$

by the arguments of 5.9.

Theorem 6.6.2. *Suppose $\zeta \in \mathbb{R}^3$ is a zero of the anisotropic indefinite integral binary Hermitian form*

$$H = \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix}, \ A, C \in \mathbb{Z}, \ B \in \mathbb{Z}[i,j], \ AC - |B|^2 < 0,$$

$$H(\zeta, 1) = A|\zeta|^2 + \overline{B}\zeta + \overline{\zeta}B + C = 0.$$

Then ζ is badly approximable as defined above. Specifically, if $|\zeta - pq^{-1}| < \epsilon$ then

$$|\zeta - pq^{-1}| \geq \frac{\mu(H)}{2|q|^2(|A|\epsilon + \sqrt{-\Delta})}$$

and

$$\liminf\{|q||\zeta q - p| : (p,q) \neq (0,0), p, q \in \mathbb{Z}[i,j,k], \ pq^{-1} \in \mathbb{R}^3\} \geq \frac{\mu(H)}{2\sqrt{-\Delta}}.$$

Proof. A version of the mean value theorem for $f : \mathbb{R}^3 \to \mathbb{R}$ is

$$f(b_1, b_2, b_3) - f(a_1, a_2, a_3) = \left(\frac{\partial f}{\partial x}(c_1, c_2, c_3), \frac{\partial f}{\partial y}(c_1, c_2, c_3), \frac{\partial f}{\partial z}(c_1, c_2, c_3)\right) \cdot (b_1 - a_1, b_2 - a_2, b_3 - a_3)$$

for some (c_1, c_2, c_3) on the line segment between (a_1, a_2, a_3) and (b_1, b_2, b_3). Applying this to the function $H(\cdot, 1)$ and using the fact that $H(\zeta, 1) = 0$ along with the Cauchy-Schwarz inequality gives

$$|H(pq^{-1}, 1)| = |H(\zeta, 1) - H(pq^{-1}, 1)| \leq |2Ax + 2B||\zeta - pq^{-1}|$$

for some x between ζ and pq^{-1}. Supposing $|\zeta - pq^{-1}| < \epsilon$ gives

$$\begin{aligned} |H(pq^{-1}, 1)| \leq 2|A||x + B/A||\zeta - pq^{-1}| &\leq 2|A|(\sqrt{-\Delta}/A + \epsilon)|\zeta - pq^{-1}| \\ &= 2(\sqrt{-\Delta} + |A|\epsilon)|\zeta - pq^{-1}|. \end{aligned}$$

Multiplication by $|q|^2$ gives

$$\mu(H) \leq |A|p|^2 + \bar{q}\overline{B}p + \bar{p}Bq + C|q|^2| \leq 2(|A|\epsilon + \sqrt{-\Delta})|q|^2, \ q \neq 0,$$

where $\mu(H) = \min\{|H(p,q)| : (p,q) \neq (0,0), \ p, q \in \mathbb{Z}[i,j,k]\}$. Hence

$$|\zeta - pq^{-1}| \geq \frac{\mu(H)}{2|q|^2(|A|\epsilon + \sqrt{-\Delta})}.$$

Letting ϵ tend to zero gives

$$\liminf\{|q||\zeta q - p| : (p,q) \neq (0,0), p, q \in \mathbb{Z}[i,j,k], \ pq^{-1} \in \mathbb{R}^3\} \geq \frac{\mu(H)}{2\sqrt{-\Delta}}.$$

$\square$

6.6.2 Experimentation and visualization for $\mathbb{Z}^3 \subseteq \mathbb{R}^3$

Here we include some experimentation and visualization.

- The number $\xi = (\sqrt{2} - 1)(1 + i + j)$ has periodic expansion

$$[0; \overline{1 - i - j, -2 - 2i - 2j, 1 - i - j, 2 + 2i + 2j}].$$

  The number $\xi + 1 + i + j$ is a zero of the Hermitian form $|u|^2 - 2|v|^2$, and the symmetry of the coefficients keeps the remainders on the union of the lines $x = y = z$ and $-x = y = z$.

- Shown in Figure 6.2 are the first 20000 remainders for $\xi = \sqrt{e/10}i + \sqrt{\pi/10}j + \sqrt{1 - (e + \pi)/10}$, which is a zero of the (isotropic, indefinite, integral) Hermitian form $|x|^2 - |y|^2$. By Theorem 6.6.1, conditional on the increasing of the denominators, the orbit should lie on finitely many spheres, which it seems to do.

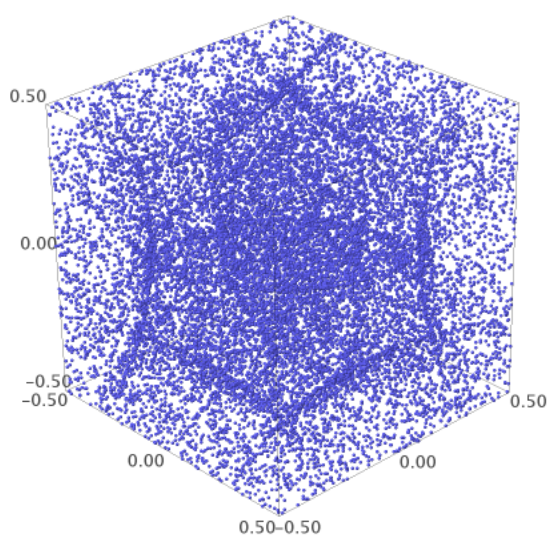

Figure 6.2: 20000 remainders for $\sqrt{e/10}i + \sqrt{\pi/10}j + \sqrt{1 - (e + \pi)/10}$ lying on a finite collection of spheres.

- Here we have remainders for $\xi = \sqrt{2}i + \sqrt{3}j$, all of which lie on the plane $x = 0$. The number ξ is a zero of $|x|^2 - 5|y|^2$, so the orbit should lie on finitely many spheres (which give circles when intersected with the plane).

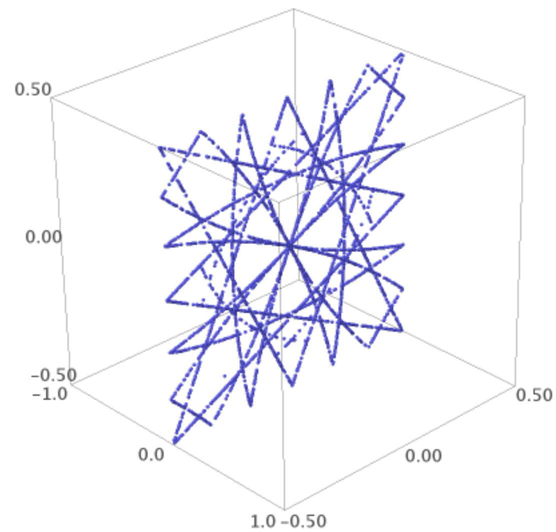

Figure 6.3: 10000 remainders for $\sqrt{2}i + \sqrt{3}j$ all lying in the plane $x = 0$.

- Figure 6.4, left, shows the orbit of $\xi = \sqrt{2} + \sqrt{3}i + \sqrt{2}j$ which, due to the symmetry of the coefficients, must lie in the planes $x = \pm z$. The number ξ is also a zero of the (anisotropic) Hermitian form $|x|^2 - 7|y|^2$ so the remainders should like on finitely many spheres (circles in the planes $x = \pm z$ here) and stay bounded away from zero. On the right in Figure 6.4 is part of the orbit of $\pi + ei + \pi j$, which also lies on the planes $x = \pm z$, but doesn't satisfy any integral Hermitian form.

6.7 Some examples of orders

Let a_i, $1 \leq i \leq n{-}1$ be positive integers, $q(v) = -\sum_i a_i v_i^2$ negative definite, and $\mathrm{Cliff}(\mathbb{R}^{n-1}, q) \subseteq \mathcal{C}_n$ the associated Clifford algebra. We have orders $\mathcal{O}_q$ coming from the integer points,

$$\mathcal{O}_q = \mathbb{Z} + \mathbb{Z}\sqrt{a_1}e_1 + \ldots + \mathbb{Z}\sqrt{a_{n-1}}e_{n-1} + \ldots + \mathbb{Z}\sqrt{a_1 \cdots a_{n-1}}e_1 \cdots e_{n-1},$$

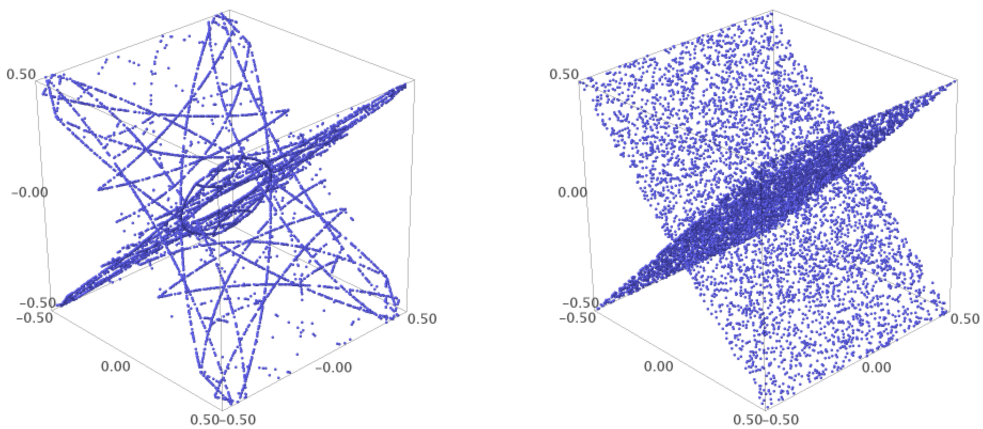

Figure 6.4: Left are 10000 remainders for $\sqrt{2}+\sqrt{3}i+\sqrt{2}j$ all lying in the planes $x = \pm z$. Compare with 10000 remainders for $\pi + ei + \pi j$ on the right.

(i.e. $\mathbb{Z}$ adjoin anti-commuting square roots of $-a_i$). We would like to identify maximal orders containing $\mathcal{O}_q$ and moreover single out those that satisfy the weak Euclidean property.

For instance, when $n = 2$, we have orders in imaginary quadratic fields and there are five orders with the weak Euclidean property (maximal orders in $\mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$). For some examples of weakly Euclidean orders when $n = 3$ and applications to sphere packings, see [She18].

In general, finding and classifying such orders seems like a difficult task (I couldn't identify the maximal order corresponding to $a_i = 1$ for all i) and one might conjecture that there are only finitely many weakly Euclidean orders in negative definite quaternion algebras over $\mathbb{Q}$ (over all dimensions).

Chapter 7

Nearest integer continued fractions over norm-Euclidean number rings and simultaneous approximation

7.1 Introduction

This chapter is an attempt to develop continued fractions in the setting of approximation in $K \otimes \mathbb{R}$, where K is a norm-Euclidean number field, i.e. a number field for which the absolute value of the field norm is a Euclidean function on $\mathcal{O}_K$.

The link between Euclidean (not necessarily for the norm!) and class number one (PID) is quite strong. Conditional on GRH, any number ring with infinitely many units is a PID if and only if it is Euclidean [Wei73]. The unconditional results are good as well, for instance: if $K/\mathbb{Q}$ is Galois and $rk(\mathcal{O}_K^\times) > 3$, then $\mathcal{O}_K$ Euclidean if and only if $\mathcal{O}_K$ is a PID [HM04]. There are number rings that are Euclidean but not norm-Euclidean, e.g. $\mathbb{Z}[\sqrt{14}]$ [Har04], but every number ring with unit rank ≥ 1 is k-stage Euclidean if and only if it has class number one (cf. [Coo76a], [Coo76b]). We also note that the S-integers of a number ring are Euclidean for large enough S [Mar75]. For a survey of the Euclidean algorithm in algebraic number fields, see [Lem95].

The absolute value of the field norm is convenient because it is multiplicative, but difficult to work with because it isn't convex, doesn't induce a metric, doesn't separate points, etc., unless $K = \mathbb{Q}$ or K is imaginary quadratic. Because of these difficulties, I could not prove much of anything about the proposed algorithm. At the end of the chapter, we discuss an accessible example, $\mathbb{Q}(\sqrt{2})$.

7.2 Simultaneous approximation over norm-Euclidean number fields via continued fractions

Let $K/\mathbb{Q}$ be an extension of degree $r+2s$, where r is the number of real embeddings $\sigma_i : K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate embeddings $\tau_i : K \hookrightarrow \mathbb{C}$. The induced map

$$\Phi : K \to \mathbb{R}^r \times \mathbb{C}^s, \ \Phi(a) = (\sigma_1(a), \ldots, \sigma_r(a), \tau_1(a), \ldots, \tau_s(a)),$$

embeds the ring of integers $\mathcal{O}_K$ as a discrete subring of $\mathbb{R}^r \times \mathbb{C}^s$ with covolume $2^{-s}\sqrt{|D_K|}$, where D_K is the field discriminant. One can ask about the approximation properties of $\Phi(K)$ inside $\mathbb{R}^r \times \mathbb{C}^s$, i.e. how well and in what manner can one approximate $X \in \mathbb{R}^r \times \mathbb{C}^s$ by elements $\Phi(a)$, $a \in K$. When $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$ (the norm-Euclidean imaginary quadratic fields), one approach is via continued fractions, which we'd like to generalize.

We view $\mathbb{R}^r \times \mathbb{C}^s$ in the boundary of $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$, the product of r copies of hyperbolic two-space and s copies of hyperbolic three-space (upper half-space model). The group $\Gamma := PGL_2(\mathcal{O}_K)$ is a finite covolume discrete subgroup of $\mathrm{Isom}((\mathcal{H}^2)^r \times (\mathcal{H}^3)^s)$, where $g \in \Gamma$ acts via the Poincaré extension of the fractional linear action on the various ideal boundaries,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ X = (x_1, \ldots, x_r, z_1, \ldots, z_s) \in \mathbb{R}^r \times \mathbb{C}^s,$$

$$g \cdot X = \left(\frac{\sigma_1(a)x_1 + \sigma_1(b)}{\sigma_1(c)x_1 + \sigma_1(d)}, \ldots, \frac{\tau_s(a)z_s + \tau_s(b)}{\tau_s(c)z_s + \tau_s(d)} \right).$$

If K is norm-Euclidean, i.e. there is a fundamental domain $\mathcal{F}$ for the action of $(\mathcal{O}_K, +)$ on $\mathbb{R}^r \times \mathbb{C}^s$ contained in the open set

$$V_1 := \{X \in \mathbb{R}^r \times \mathbb{C}^s : |x_1 \cdots x_r||z_1 \cdots z_s|^2 < 1\},$$

then we can uniquely write $X = \lfloor X \rfloor + \{X\}$ with $\lfloor X \rfloor \in \Phi(\mathcal{O}_K)$ and $\{X\} \in \mathcal{F}$ and try to write X as a continued fraction

$$X = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \ldots]$$

where $a_0 = \{X\}$, $a_1 = \lfloor 1/X \rfloor$, etc., the norm-Euclidean condition ensuring that $\mathcal{F}^{-1} = \{1/X : X \in \mathcal{F}\}$ is contained in the complement of V_1. If $\mathcal{F} \subseteq V_\epsilon = \{X \in \mathbb{R}^r \times \mathbb{C}^s : |x_1 \cdots x_r||z_1 \cdots z_s|^2 < \epsilon\}$, $0 < \epsilon < 1$, then $\mathcal{F}^{-1}$ is in the complement of $V_{1/\epsilon}$ with $1/\epsilon > 1$. We use the notation

$$N(X) = |x_1| \cdots |z_s|^2$$

for the absolute value of the extension of the field norm below. The algorithm and notation are as follows:

$$X_0 = X - a_0,\ a_0 = \lfloor X \rfloor,\ \frac{1}{X_n} = a_{n+1} + X_{n+1},\ X_{n+1} = \left\{ \frac{1}{X_n} \right\},\ a_n = \left\lfloor \frac{1}{X_n} \right\rfloor,$$

$$X_n = T^n(X),\ T : \mathcal{F} \to \mathcal{F},\ T(X) = \left\{ \frac{1}{X} \right\}.$$

The convergents $p_n/q_n \in K$ to X, $p_n, q_n \in \mathcal{O}_K$, are defined recursively by

$$p_{-1} = 1,\ q_{-1} = 0,\ p_n = a_n p_{n-1} + p_{n-2},\ q_n = a_n q_{n-1} + q_{n-2},$$

i.e.

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Here and elsewhere we abuse notation, ignoring the distinction between $a \in K$ and $a \in \Phi(K)$ whenever convenient. We introduce the notation

$$X = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

although we are currently unable to show that the right hand side exists analytically.

The following lemma lists some purely algebraic properties of the continued fraction algorithm.

Lemma 7.2.1. *For $X \in \mathbb{R}^r \times \mathbb{C}^s \setminus K^{r+s}$ we have*

$$q_n \neq 0,\ (X_n)_i \neq 0 \text{ for } 1 \leq i \leq r+s,\ a_n \neq 0 \text{ for } n \geq 1,$$

$$X = \frac{p_n + X_n p_{n-1}}{q_n + X_n q_{n-1}},\ q_n X_n - p_n = (-1)^n X_0 \cdots X_n,\ X - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2 (1/X_n + q_{n-1}/q_n)},$$

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-1}}{q_{n-2}} = a_n + \cfrac{1}{a_{n-1} + \frac{1}{\cdots + \frac{1}{a_1}}} \overset{alg.}{=} [a_n; a_{n-1}, \ldots, a_1],$$

where by the overset "alg." we emphasize that the algorithm, when applied to the rational q_n/q_{n-1}, does not necessarily produce the continued fraction on the right.

The following lemma gives convergence with respect to the extension of the field norm.

Lemma 7.2.2. *For $X \in \mathbb{R}^r \times \mathbb{C}^s \setminus K^{r+s}$ we have*

$$\lim_{n\to\infty} N(X - p_n/q_n) = 0.$$

Proof. We have

$$N(X - p_n/q_n) = \frac{N(X_0 \cdots X_n)}{N(q_n)} \le N(X_0 \cdots X_n) \le \epsilon^{n+1} \to 0.$$

□

7.3 Experimentation in $\mathbb{Q}(\sqrt{2})$

The simplest example that comes to mind is shown in Figure 7.1 for the norm-Euclidean field $\mathbb{Q}(\sqrt{2})$. The unit hyperbola $N(x_1, x_2) = 1$ is shown in red, a fundamental domain for the additve group of the integers on the plane is shown as a blue rectangle inside the unit hyperbola, the integers $\mathbb{Z}[\sqrt{2}]$ are in green, and the inversion of the fundamental domain is also shown in blue outside the unit hyperbola. The restriction of the algorithm to the diagonal (x, x) gives the usual nearest integer continued fraction algorithm (considered by Hurwitz [Hur89] and generalized by Nakada to α-continued fractions [Nak81]). It seems that the norms of the denominators are not necessarily increasing for this algorithm, but they "mostly" do so as shown in Figure 7.2 (i.e. the dots mostly seem to stay within the unit hyperbola). The table below shows relevant information for a somewhat random vector ("somewhat" because it was chosen to have non-increasing denominators early on)

$$\begin{aligned}(x_1, x_2) &= (-0.164685\ldots, 0.347722\ldots)\\ &\text{“}=\text{”}\ [0; -2 - 3\sqrt{2}, 4 + 2\sqrt{2}, 1 - \sqrt{2}, 12 - 10\sqrt{2}, -1 + \sqrt{2}, \ldots]\end{aligned}$$

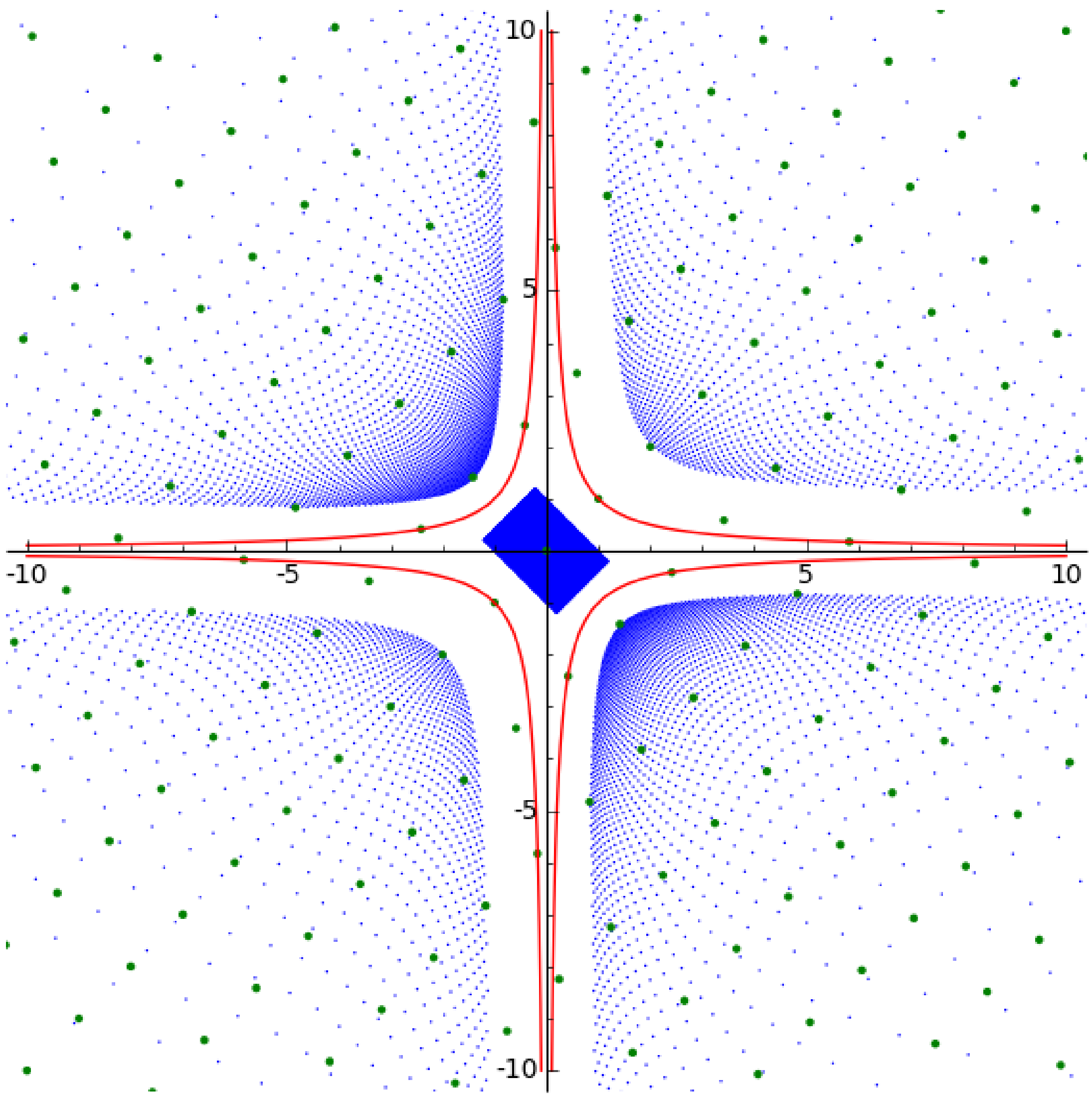

Figure 7.1: Setup for nearest integer continued fractions over $\mathbb{Q}(\sqrt{2})$: $\mathcal{F}$, $\mathcal{F}^{-1}$ blue, $\mathbb{Z}[\sqrt{2}]$ green dots, $N(x) = 1$ red.

For all computations I've done, the algorithm seems to converge quickly (as it does for the example in the table).

n	1	2	3	4	5	6
a_n	0	$-2-3\sqrt{2}$	$4+2\sqrt{2}$	$1-\sqrt{2}$	$12-10\sqrt{2}$	$-1+2\sqrt{2}$
$\frac{p_n}{q_n}$	$\frac{0}{1}$	$\frac{1}{-2-3\sqrt{2}}$	$\frac{4+2\sqrt{2}}{-19-16\sqrt{2}}$	$\frac{1-2\sqrt{2}}{11}$	$\frac{56-32\sqrt{2}}{113-126\sqrt{2}}$	$\frac{-183+142\sqrt{2}}{-606+352\sqrt{2}}$
$\frac{p_n}{q_n} \approx$	0	$(-0.160, 0.445)$	$(-0.164, 0.322)$	$(-0.166, 0.348)$	$(-0.164, 0.347)$	$(-0.164, 0.347)$
$\lvert N(q_n)\rvert$	1	12	151	121	11711	119428

Finally, we discuss periodicity for quadratic irrationals. For example, the vector

$$\left(\sqrt{3+\sqrt{2}}, \sqrt{3-\sqrt{2}}\right) = (2.1010\ldots, 1.2592\ldots)$$

is a root of the totally indefinite anisotropic rational quadratic form $Q(x, y) = x^2 - (3 + \sqrt{2})y^2$ and is therefore badly approximable as shown in 5.6.3 and 5.9. As expected, the continued fraction expansion of the above vector seems to be eventually periodic

$$\left(\sqrt{3+\sqrt{2}}, \sqrt{3-\sqrt{2}}\right) \text{ `` = ''} [2; \overline{4+4\sqrt{2}, 4}].$$

and convergent, approximations to the first few convergents being

$$(2, 2),\ (2.103, 1.396),\ (2.100, 1.289),\ (2.101, 1.266),\ (2.101, 1.260),\ (2.101, 1.259).$$

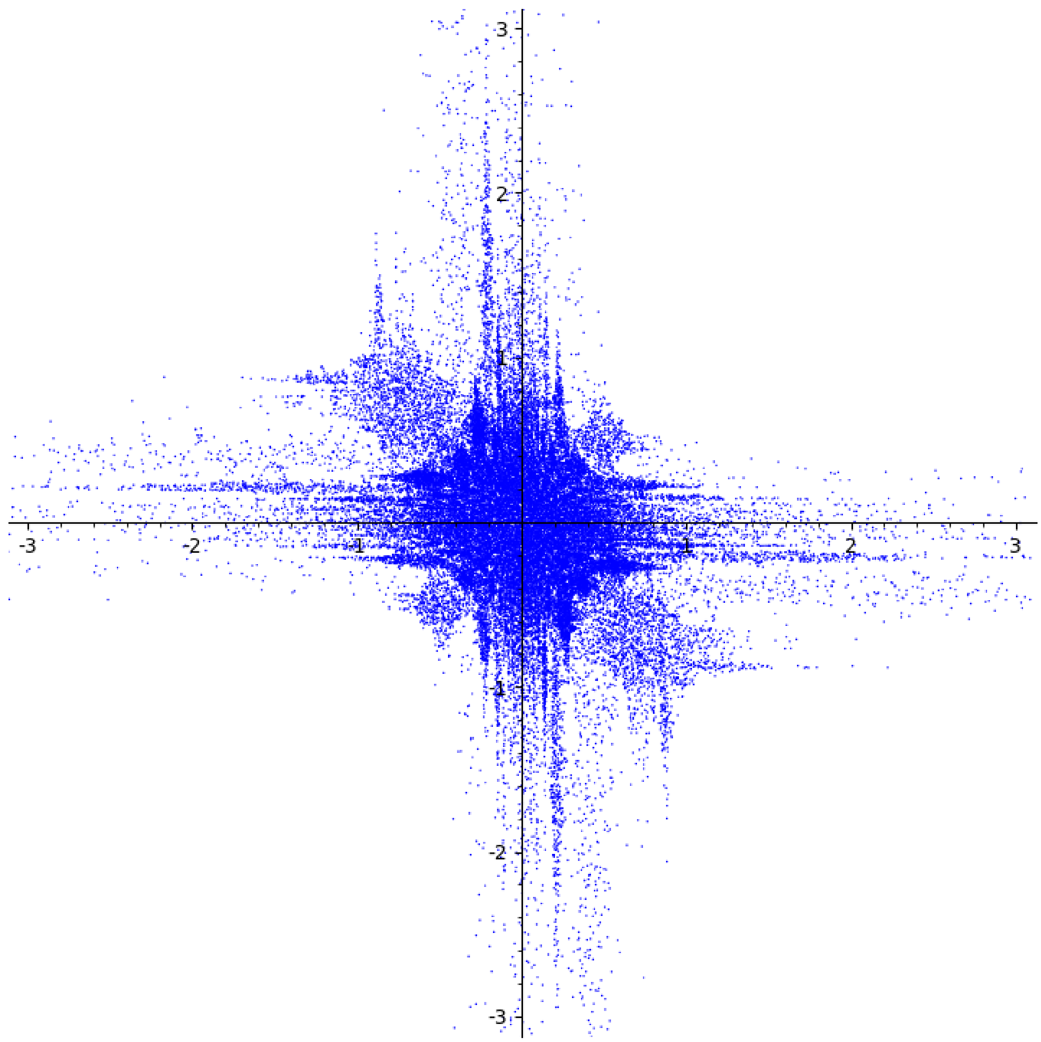

Figure 7.2: Ratio of the denominators q_n/q_{n+1} for 5000 random numbers and $1 \leq n \leq 10$.

Chapter 8

Miscellaneous results on the discrete Markoff spectrum

8.1 Introduction

In this chapter we give a quick overview of the discrete Markoff spectrum and give some identities for sums over Markoff numbers. Unfortunately for me, these sums are special cases of Mcshane's identity for the following sum over the lengths of simple closed geodesics on any once-punctured torus (with complete hyperbolic metric):

$$\sum_{\gamma} \frac{1}{1+e^{l(\gamma)}} = \frac{1}{2}.$$

Moreover a proof of Mcshane's identity along the lines of what is presented below was already given in [Bow96].

We also prove (although this is basically an observation I hadn't seen elsewhere) that the limits of the roots of Markoff forms down non-trivial paths in the associated tree of forms are transcendental. This follows from the fact that the continued fraction expansion of such numbers begin with arbitrarily large repeated words (so-called "stammering" continued fractions) and a result of [AB05] (an application of the Subspace Theorem of W. M. Schmidt). These numbers are badly approximable (with only ones and twos in their continued fraction expansion) and well-approximated by quadratic irrationals (the roots of the Markoff forms). Geometrically, they correspond to some of the simple non-closed geodesics maximally distant from the cusp of the once-punctured torus associated to the commutator subgroup of $SL_2(\mathbb{Z})$. These transcendental numbers are also given as infinite sums of rational numbers that can be written explicitly in terms of the path taken to construct them (in the tree of Markoff numbers). As an example, the number

$$\sum_{i=0}^{\infty}(-1)^{i+1}\left(3-\frac{m_{i+2}}{m_i m_{i+1}}\right),\ m_{i+3}=3m_{i+2}m_{i+1}-m_i,\ (m_1,m_2,m_3)=(5,13,194),$$

is transcendental, corresponding to a periodic path towards $\frac{3-\sqrt{5}}{2}$.

References for this chapter include: [CF89], [Aig13], [Cas57], [Mar79], [Mar80], [PM93], [Fro13], [Coh55], [Sch76].

8.2 Markoff numbers and forms, Frobenius coordinates, and Cohn matrices

Let

$$A=M_{\frac{0}{1}}=\begin{pmatrix}1&2\\2&5\end{pmatrix},\ B=M_{\frac{1}{1}}=\begin{pmatrix}0&-1\\1&3\end{pmatrix},$$

and recursively define $M_{\frac{p+p'}{q+q'}}=M_{\frac{p}{q}}M_{\frac{p'}{q'}}$ for $p/q<p'/q'$ and $pq'-p'q=-1$, associating an element of $SL_2(\mathbb{Z})$ to each number $\mathbb{Q}\cap[0,1]$. Define $m_{\frac{p}{q}}:=\frac{1}{3}\mathrm{tr}M_{\frac{p}{q}}$ and let $\mathcal{M}=\{m_{\frac{p}{q}}:p/q\in\mathbb{Q}\cap[0,1]\}$ be the *multiset* of these values. For each $M_{\frac{p}{q}}$ define the indefinite integral binary quadratic form

$$f_{\frac{p}{q}}(x,y)=cx^2+(d-a)xy-by^2,\ M_{\frac{p}{q}}=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

The $f_{\frac{p}{q}}$ are fixed-point forms of the $M_{\frac{p}{q}}$. For three adjacent rationals in $[0,1]$,

$$\frac{p}{q}<\frac{p''}{q''}=\frac{p+p'}{q+q'}<\frac{p'}{q'},\ pq'-p'q=-1,$$

we have

$$m_{\frac{p}{q}}^2+m_{\frac{p'}{q'}}^2+m_{\frac{p''}{q''}}^2=3m_{\frac{p}{q}}m_{\frac{p'}{q'}}m_{\frac{p''}{q''}},$$

and every unordered positive triple of integer solutions to the Markoff equation $x^2+y^2+z^2=3xyz$ except $(1,1,1)$ and $(1,1,2)$ occurs in this fashion.

For any solution $m\geq n_1,n_2$ of the Markoff equation, let u be the smallest non-negative solution to

$$u\equiv\pm n_1/n_2 \bmod m,$$

and let $v = \frac{u^2+1}{m}$. Then the matrices $M_{\frac{p}{q}}$ and the fixed-point forms $f_{\frac{p}{q}}$ satisfy (with $m = m_{\frac{p}{q}}$)

$$M_{\frac{p}{q}} = \begin{pmatrix} u & 3u-v \\ m & 3m-u \end{pmatrix}, \; f_{\frac{p}{q}}(x,y) = mx^2 + (3m-2u)xy + (v-3u)y^2.$$

The Markoff forms and Markoff equation were introduced by Markoff to characterize the indefinite binary quadratic forms $f(x,y) = ax^2 + bxy + cy^2$ (with real coefficients) satisfying

$$\frac{\mu_f}{\sqrt{\Delta_f}} > \frac{1}{3}, \; \mu_f = \inf\{|f(x,y)| : (x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}\}, \; \Delta_f = b^2 - 4ac.$$

A version of the coordinates p/q for Markoff numbers was introduced by Frobenius [Fro13]. The use of the matrices above is due to Cohn [Coh55]. A natural choice of generators (motivated by Markoff's characterization in terms of continued fractions) is

$$\widetilde{A} = 2 + \frac{1}{2+\frac{1}{x}} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \; \widetilde{B} = 1 + \frac{1}{1+\frac{1}{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The matrices A, B, are conjugate to $\widetilde{A}$, $\widetilde{B}$, shifting the roots of the fixed point forms by -2 (so that the "first root" lies in the interval $[0,1]$). In any case, any relevant pair generates the commutator subgroup Γ' of $\Gamma = SL_2(\mathbb{Z})$, with quotient $\Gamma' \backslash \mathcal{H}^2$ the maximally symmetric once-punctured torus.

Markoff [Mar79], [Mar80] showed the following.

Theorem 8.2.1. *The set of values*

$$(1/3,\infty) \cap \left\{ \frac{\mu_f}{\sqrt{\Delta_f}} : f = ax^2 + bxy + cy^2, \; a,b,c \in \mathbb{R}, \; \Delta_f > 0 \right\} = \left\{ \frac{1}{\sqrt{5}}, \; \frac{1}{\sqrt{2}}, \; \frac{5}{\sqrt{221}}, \dots \right\}$$

is discrete with $1/3$ *as its only limit point. Each form contributing is* $SL_2(\mathbb{Z})$*-equivalent to exactly one of the Markoff forms* $f_{\frac{p}{q}}$.

Geometrically, the Markoff forms correspond to the "minimal height" geodesics in the modular surface (the number $\frac{\sqrt{\Delta_f}}{2|a|}$ is the y-coordinate of the geodesic of the top of the semi-circle connecting the roots of the quadratic form f). Analytically, the Lagrange spectrum is the set of values

$$\liminf_{q\to\infty} \{|q(q\alpha - p)|\}, \; \alpha \in \mathbb{R},$$

and the roots of the Markoff forms above give representative α for the smallest values of the Lagrange spectrum (i.e. the discrete set of values less than 3).

For proofs (and much more), see [Aig13], [CF89], [Bom07], or [Cas57]. For geometric/topological discussions and results, see [Riv12], [CDG$^+$98], [Ser85], [She85], [Spr17].

8.3 A variation of Mcshane's identity

Proposition 8.3.1. *The function (see Figure 8.3)*

$$\phi : \mathbb{Q} \cap [0,1] \to [0,1],\ \ p/q \mapsto u/m$$

is decreasing with

$$\lim_{r \to p/q^-} \phi(r) - \phi(p/q) = \phi(p/q) - \lim_{r \to p/q^+} \phi(r) = \frac{3}{2} - \frac{\sqrt{9m^2-4}}{2m}.$$

Consequently,

$$\sum_{m \in \mathcal{M}} 3 - \frac{\sqrt{9m^2-4}}{m} = \frac{7-\sqrt{5}-\sqrt{8}}{2}.$$

Proof. Suppose

$$\frac{p}{q} < \frac{p''}{q''} = \frac{p+p'}{q+q'} < \frac{p'}{q'},\ \ pq' - p'q = -1,$$

with corresponding Markoff numbers m, m', and m''. Multiplying the Cohn matrices

$$\begin{pmatrix} u & 3u-v \\ m & 3m-u \end{pmatrix} \begin{pmatrix} u' & 3u'-v' \\ m' & 3m'-u' \end{pmatrix} = \begin{pmatrix} u'' & 3u''-v'' \\ m'' & 3m''-u'' \end{pmatrix}$$

gives

$$u'' = uu' + 3um' - \frac{u^2+1}{m}m',\ \ m'' = mu' - um' + 3mm'.$$

The formula for m'' above gives

$$\frac{u}{m} - \frac{u'}{m'} = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{1}{m^2} - \frac{1}{(m')^2}} = 3 - \frac{m''}{mm'},$$

and letting m or m' to infinity gives the limits in the Proposition. $\square$

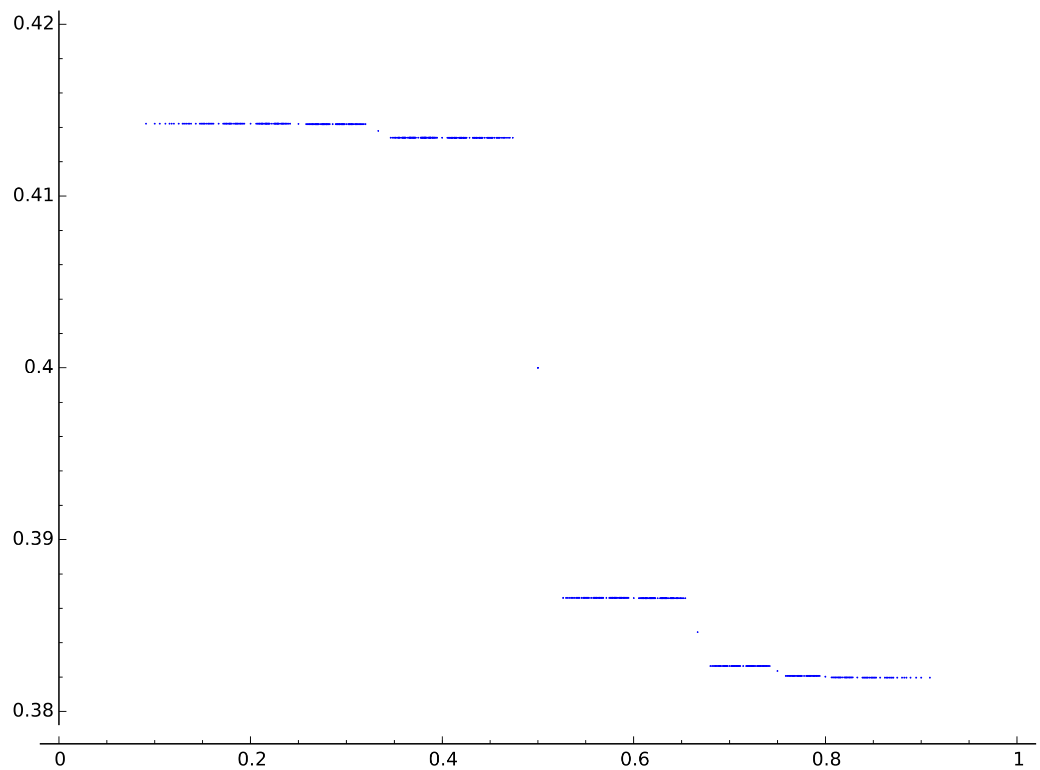

Figure 8.1: A plot of $(p/q, u/m)$ at depth ≤ 10 in the Farey tree (excluding $(0, 1/2)$ and $(1, 0)$).

In order to relate this to Mcshane's identity, we note that the length of the geodesic associated to a primitive hyperbolic element g of $PSL_2(\mathbb{R})$ is

$$l(g) = 2 \text{ arccosh}\left(\frac{\text{tr}(g)}{2}\right),$$

so that

$$\frac{1}{1+e^l} = 1 - \frac{\sqrt{t^2-1}}{t},\ t = \text{tr}(g),\ l = l(g).$$

For the Cohn matrices, we have $\text{tr}(g) = 3m$ so that

$$\frac{1}{1+e^l} = \frac{1}{3}\left(3 - \frac{\sqrt{9m^2-4}}{m}\right).$$

The sum in the proposition above does not sum over all of the simple geodesics of the once-punctured torus associated with the commutator subgroup of $SL_2(\mathbb{Z})$. See [Bow96] for a proof of Mcshane's identity using the tree structure associated to the simple closed geodesics of a once punctured torus.

8.4 Transcendence of limits of Markoff forms

The normalized Markoff forms

$$x^2 + \left(3 - 2\frac{u}{m}\right)xy + \left(\frac{u^2}{m^2} - 3\frac{u}{m} + \frac{1}{m^2}\right)y^2$$

tend, as $u/m \to \alpha$, to the form

$$x^2 + (3-2\alpha)xy + \alpha(\alpha-3)y^2 = (x-\alpha y)(x-(\alpha-3)y)$$

which has discriminant 9, absolute minimum 1, and roots α, $\alpha - 3$. Any such α is badly approximable, with partial quotients in $\{1, 2\}$. The "trivial" paths in the Farey tree (i.e. those representing rational numbers) give nothing new. The left and right limits give the forms with

$$\alpha = \frac{u}{m} \pm \left(\frac{3}{2} - \frac{\sqrt{9m^2-4}}{2m}\right)$$

whose roots are quadratic irrationals equivalent to the roots of the Markoff forms themselves. However, all of the other limiting forms give *transcendental* α. This is a consequence of [AB05], Theorem 1, a special case of which we give below, which in turn follows from the Subspace Theorem.

Theorem 8.4.1 (cf. [AB05], Theorem 1, with $w = 2$). *Let $\alpha = [0; a_1, a_2, \ldots]$ and suppose there are arbitrarily large values of n such that*

$$a_i = a_{n+i}, \; 1 \leq i \leq n,$$

i.e. the sequence of partial quotients has longer and longer prefixes that are "doubled." If α is not quadratic (the sequence of partial quotients is not eventually periodic), then α is transcendental.

Proposition 8.4.2. *Let $\theta \in [0,1]$ be irrational and let $\alpha = \lim_{r \to \theta} \phi(r)$ be a limiting value of u/m down a non-trivial path in the Farey tree. Then α is transcendental.*

Proof. The conditions of the previous theorem are satisfied along any non-trivial path via the "concatenation" rules for the continued fraction expansion of the roots of the Markoff forms. Every left going down the Farey tree doubles a prefix in the continued fraction expansion; see Figure 8.4. These prefixes will grow as long as there are infinitely many right and left turns (i.e. we follow a non-trivial path). □

For instance, following the path to $\theta = \frac{3-\sqrt{5}}{2}$ gives

$$\alpha = \frac{2}{5} + \sum_{i=0}^{\infty} (-1)^{i+1} \left(3 - \frac{m_{i+2}}{m_i m_{i+1}} \right) = 0.38658908134980549908\ldots,$$

$$2 + \alpha = [2_2, 1_2, 2_2, 1_4, 2_2, 1_2, 2_2, 1_4, 2_2, 1_4, 2_2, 1_2, 2_2, 1_4, 2_2, 1_2, 2_2, 1_4, 2_2, 1_4 \ldots],$$

where the subscript indicates repeated digits and m_i are the Markoff numbers in the recurrence

$$m_{i+3} = 3m_{i+2}m_{i+1} - m_i, \; (m_1, m_2, m_3) = (5, 13, 194).$$

More generally, the transcendental α above can be written as

$$\frac{2}{5} + \sum_{i=0}^{\infty} \epsilon_i \left(3 - \frac{m_{i+2}}{m_i m_{i+1}} \right)$$

with $\epsilon_i = \pm 1$ according one turns right $(-)$ or left $(+)$ down the Farey tree (starting at $p/q = 1/2$) and (m_i, m_{i+1}, m_{i+2}) is the corresponding Markoff triple.

These "irrational slopes" were studied in [Coh74], and a description for all forms with $\mu/\sqrt{\Delta} = 1/3$ was determined in [Yas98]. The Markoff forms correspond to simple closed geodesics on a once-punctured torus and the limiting forms to some of the simple open geodesics. The closed geodesics with one self-intersection also give an interesting subset of the Markoff spectrum, a sequence with $\mu/\sqrt{\Delta}$ increasing to 1/3 [CM93].

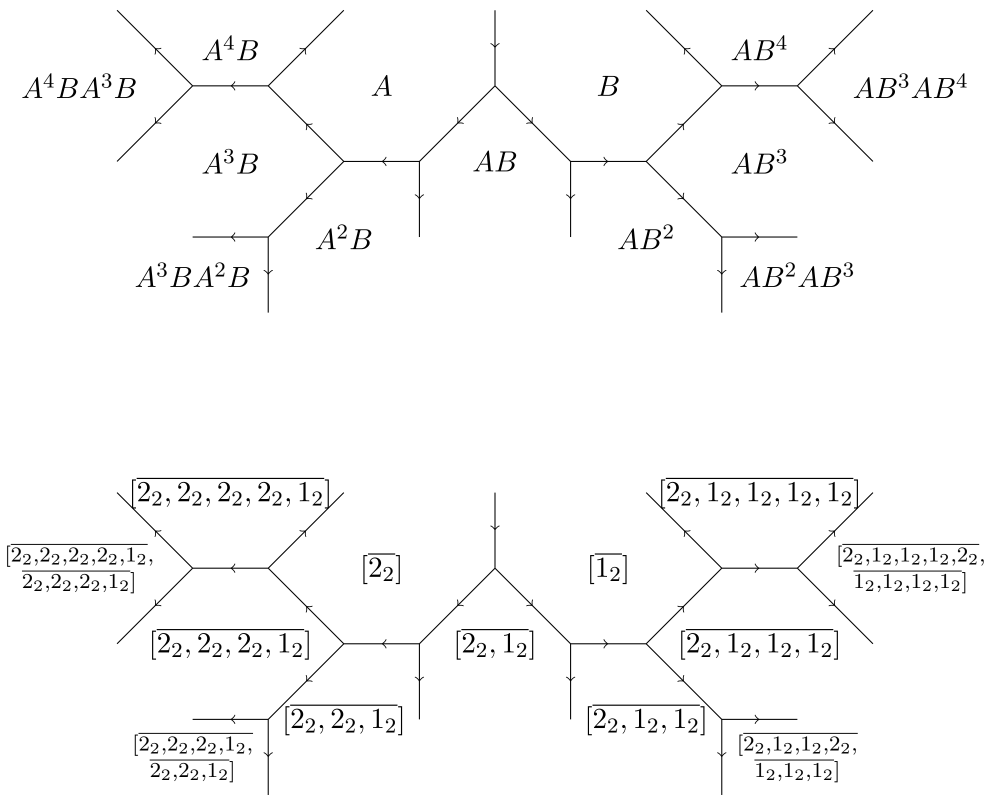

Figure 8.2: Top: Part of the "half-topograph" (or tree of triples) of words in A and B giving Markoff forms (the fixed point forms) and Markoff numbers (one-third of the trace). Bottom: The continued fraction expansion of the roots of the fixed point forms from the $\widetilde{A}$, $\widetilde{B}$ tree.

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