Applications of hyperbolic geometry to continued fractions and Diophantine approximation

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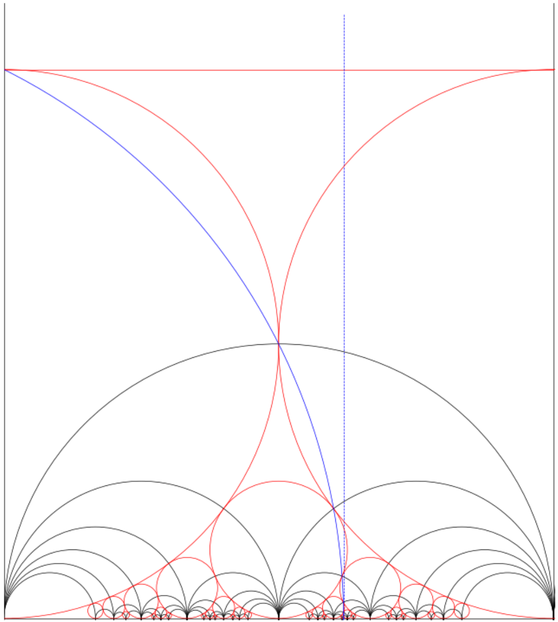
A picture



Our goal is to generalize features of the preceding picture to some nearby settings:

$\mathcal{H}^2$	$\mathcal{H}^3$	$(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$	$\mathcal{H}^n$
$P^1(\mathbb{R})$	$P^1(\mathbb{C})$	$P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$	S^{n-1}
$SL_2(\mathbb{R})$	$SL_2(\mathbb{C})$	$SL_2(F \otimes \mathbb{R})$	$SV_{n-1}(\mathbb{R})$
$SL_2(\mathbb{Z})$	$SL_2(\mathcal{O})$	”	$SV(\mathcal{O})$
ideal triangles	right-angled ideal polyhedra		
horoball neighborhoods	bounded geodesic trajectories	”	
quad. forms	quad./Herm. forms	”	”
closed geodesics	closed surfaces	aniso. subgroups	”

Ingredients

Hyperbolic two-space:

$$\begin{aligned} \mathcal{H}^2 &= \{z = x + iy \in \mathbb{C} : y > 0\}, \\ \partial\mathcal{H}^2 &= P^1(\mathbb{R}), \\ \mathrm{Isom}(\mathcal{H}^2) &= PGL_2(\mathbb{R}), \end{aligned}$$

$$g \cdot z = \frac{az+b}{cz+d},\ \frac{a\bar{z}+b}{c\bar{z}+d}\ (\det g = \pm 1),$$

$$\mathrm{Stab}^+(i) = SO_2(\mathbb{R})/\{\pm 1\} \cong SO_2(\mathbb{R}).$$

Hyperbolic three-space:

$$\begin{aligned} \mathcal{H}^3 &= \{\zeta = z + jt \in \mathbb{H} : t > 0, z \in \mathbb{C}\}, \\ \partial\mathcal{H}^3 &= P^1(\mathbb{C}), \\ \mathrm{Isom}(\mathcal{H}^3) &= PSL_2(\mathbb{C}) \rtimes \langle \tau \rangle, \end{aligned}$$

$$g \cdot \zeta = (a\zeta + b)(c\zeta + d)^{-1},\ \tau(\zeta) = \bar{z} + jt,$$

$$\mathrm{Stab}^+(j) = SU_2(\mathbb{C})/\{\pm 1\} \cong SO_3(\mathbb{R}).$$

Hyperbolic two- and three-space are the Riemannian symmetric spaces associated to $G = SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$. The points can be identified with roots of binary forms:

$$SL_2(\mathbb{R})/SO_2(\mathbb{R}) \to \{\text{det. 1 pos. def. bin. quadratic forms}\} \to \mathcal{H}^2$$

$$g \mapsto gg^t = ax^2 + bxy + cy^2 = Q \mapsto \frac{-b + \sqrt{b^2 - 4ac}}{2a} =: Z(Q)$$

$$SL_2(\mathbb{C})/SU_2(\mathbb{C}) \to \{\text{det. 1 pos. def. bin. Hermitian forms}\} \to \mathcal{H}^3$$

$$g \mapsto gg^* = az\bar{z} + b\bar{z}w + \bar{b}z\bar{w} + cw\bar{w} = H \mapsto \frac{-b + j\sqrt{ac - b\bar{b}}}{a} =: Z(H)$$

We have

$$Z(Q^g) = g^{-1} \cdot Z(Q), \; Z(H^g) = g^{-1} \cdot Z(H),$$

where the actions above are given by linear change of variable ^{g} and Möbius transformations $g\cdot$.

Indefinite forms parameterize codimension one geodesic subspaces:

$$Q = ax^2 + bxy + cy^2 \;\; (b^2 - 4ac > 0)$$
$$\sim Z(Q) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\sim \text{geodesic between the roots,}$$

$$H = az\bar{z} + b\bar{z}w + \bar{b}z\bar{w} + cw\bar{w} \;\; (ac - |b|^2 < 0)$$
$$\sim Z(H) = \left\{ z : |z + b/a|^2 = \frac{|b|^2 - ac}{a^2} \right\}$$
$$\sim \text{geodesic plane with boundary } Z(H).$$

Once again the association is SL_2-equivariant:

$$Z(Q^g) = g^{-1} \cdot Z(Q), \;\; Z(H^g) = g^{-1} \cdot Z(H).$$

We can identify $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ with spin bundles over $\mathcal{H}^2$ and $\mathcal{H}^3$, or $PSL_2(\mathbb{R})$ and $PSL_2(\mathbb{C})$ with the unit tangent bundle of $\mathcal{H}^2$ and the oriented orthonormal frame bundle of $\mathcal{H}^3$. Concretely (fixing base points (i,i) and $(j,\{1,i,j\})$ respectively and letting $v \in \mathbb{C}$, $\eta \in \mathbb{R}+\mathbb{R}i+\mathbb{R}j$ be unit tangent vectors), we have the derivative action

$$g \cdot (z,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,v) = \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right) = \left(g\cdot z, \frac{dg}{dz}\cdot v\right),$$
$$g \cdot (\zeta,\eta) = (g(\zeta), (\zeta c+d)^{-1}\eta(c\zeta+d)^{-1}).$$

In this setup, the geodesic flow and frame flow are given by

$$t \mapsto g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Given a number field F or signature (r, s) we have the $\mathbb{R}$-algbra

$$F \otimes \mathbb{R} = \frac{\mathbb{Q}[x]}{(m(x))} \otimes \mathbb{R} = \prod_{\substack{\sigma \\ \text{real}}} \frac{\mathbb{R}[x]}{(x - \sigma(\alpha))} \prod_{\substack{\{\sigma,\bar{\sigma}\} \\ \text{complex}}} \frac{\mathbb{R}[x]}{(x^2 - (\sigma(\alpha) + \bar{\sigma}(\alpha))x + \sigma(\alpha)\bar{\sigma}(\alpha))}$$

$$\cong \mathbb{R}^r \times \mathbb{C}^s,$$

and the ring of integers $\mathcal{O}$ is a discrete subring. The group

$$\Gamma = SL_2(\mathcal{O}) \subseteq SL_2(F \otimes \mathbb{R}) = G$$

is a non-uniform lattice (non-cocompact, finite covolume discrete subgroup), and we will be interested in the symmetric and locally symmetric spaces

$$(\mathcal{H}^2)^r \times (\mathcal{H}^3)^2 = G/K, \ \Gamma\backslash G/K \ (K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s).$$

For instance, when there is only one Archimedean place, we have the modular surface and the Bianchi orbifolds:

$$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}), \ SL_2(\mathcal{O})\backslash SL_2(\mathbb{C})/SU_2(\mathbb{C})$$

(where $\mathcal{O}$ is the ring of integers in an imaginary quadratic field). One can ask about approximating elements of $F \otimes \mathbb{R}$ by elements of F.

Given $g \in SL_n(\mathbb{R})$, let $\Lambda_g \subseteq \mathbb{R}^n$ be the $\mathbb{Z}$-span of the rows of g. (Oriented) change of basis corresponds to the coset $SL_n(\mathbb{Z})g$. The basic compactness criterion in the space of unimodular lattices $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ is the following.

Mahler's criterion

$X \subseteq SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ is precompact iff the lengths of all non-zero vectors in the corresponding unimodular lattices $\Lambda_x, x \in X$ are uniformly bounded below.

Badly approximable systems of linear forms	$\Longleftrightarrow$	Bounded geodesic trajectories in the space of lattices

For example:

$\xi \in \mathbb{R}$ is badly approximable, i.e. there exists $C > 0$ such that

$$|\xi - p/q| \geq C/q^2, \; p/q \in \mathbb{Q},$$

if and only if

$$SL_2(\mathbb{Z}) \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} SO_2(\mathbb{R}), \; t \geq 0,$$

is bounded.

Example: $[G = SL_2(\mathbb{R}),\ \Gamma = SL_2(\mathbb{Z}).]$ Let Q be an integral binay quadratic form and $SO(Q,\mathbb{R}) \subseteq G$ the group of g such that $g^t Q g = Q$.

If $Q(x,y) \neq 0$ for all $x, y \in \mathbb{Z}$, then $SO(Q,\mathbb{Z})\backslash SO(Q,\mathbb{R})$ is compact in $\Gamma\backslash G$.

This follows from Mahler's criterion since $\|(x,y)g^t\|$ cannot be arbitrarily small for $(0,0) \neq (x,y) \in \mathbb{Z}^2$:

$$1 \leq \left|(x,y)Q\begin{pmatrix} x \\ y \end{pmatrix}\right| = \left|(x,y)g^t Q g\begin{pmatrix} x \\ y \end{pmatrix}\right|.$$

Right-translating such quotients (i.e. changing basepoint) gives compact totally geodesic subspaces of $\Gamma\backslash G/K$ (closed geodesics on the modular surface in the example above).

Generally speaking, $\mathrm{Isom}^+(\mathcal{H}^n) = O(1,n)^\circ$, but we can cram this information into two-by-two matrices with entries in the definite Clifford algebra

$$\mathbb{R}[e_1, \ldots, e_{n-2}],\ e_i^2 = -1,\ e_i e_j = -e_j e_i.$$

We identify $\mathbb{R}^{n-1}$ with the "paravectors" $\mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_{n-2}\}$. The orientation preserving isometries can be represented by the group

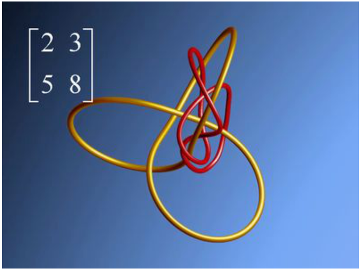
$$SV_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ac^*, bd^* \in \mathbb{R}^{n-1}, ad^* - bc^* = 1 \right\}.$$

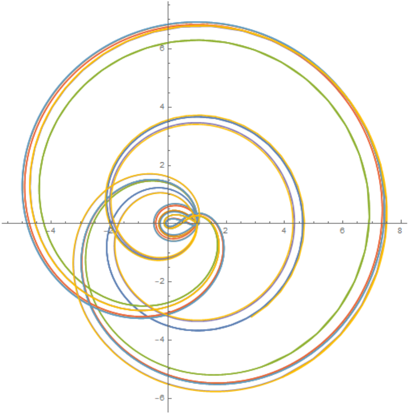
Here $*$ is the "reversal" involution induced by

$$(e_{i_1} \cdots e_{i_k})^* = e_{i_k} \cdots e_{i_1} = (-1)^{k(k-1)/2} e_{i_1} \cdots e_{i_k}.$$

[When $n = 2$, 3, we get $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.]

We now present some aspects of simple continued fractions, with an emphasis on badly approximable numbers (especially quadratic irrationals).





The Euclidean algorthim [iterating $(a, b) \mapsto (b, a \bmod b)$]

$$\begin{aligned} a &= b a_0 + r_0, \ \ 0 \le r_0 < b \\ b &= r_0 a_1 + r_1, \ \ 0 \le r_1 < r_0 \\ r_0 &= r_1 a_2 + r_2, \ \ 0 \le r_2 < r_1 \\ &\ldots, \end{aligned}$$

or written in matrices

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ r_0 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &\ldots, \end{aligned}$$

expresses a rational number a/b as a finite continued fraction

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots \frac{1}{a_n}}}} =: [a_0; a_1, \ldots, a_n].$$

Extending this to irrational numbers $\xi = \lfloor \xi \rfloor + \{\xi\} = a_0 + \xi_0$ gives a dynamical system

$$T : (0,1) \to (0,1),\ \xi_0 \mapsto \{1/\xi_0\}$$

and infinite sequences

$$\xi_{n+1} = \{1/\xi_n\} = T^{n+1}\xi_0,\ a_{n+1} = \lfloor 1/\xi_n \rfloor = \left\lfloor \frac{1}{T^n \xi_0} \right\rfloor$$

with

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} =: [a_0; a_1, a_2, \ldots].$$

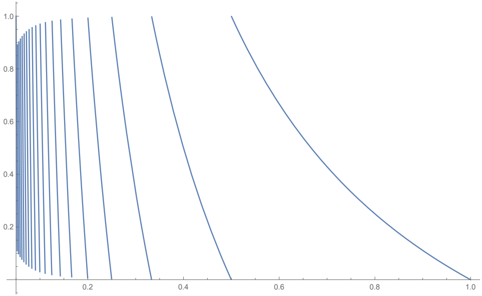
Stopping after n iterations gives rational approximations p_n/q_n to ξ, where

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

The branches of T^{-1} are all surjective and we have bijections

$$\mathbb{R} \setminus \mathbb{Q} \cong \mathbb{Z} \times \mathbb{N}^{\mathbb{N}}, \ \mathbb{Q} = \{[a_0; a_1, \ldots, a_n] : n \geq 0, \ a_n \neq 1 \text{ if } n \geq 1\}.$$

T is the left shift on these sequences, $T([0; a_1, a_2, \ldots]) = [0; a_2, a_3, \ldots]$.



The convergents p_n/q_n to ξ have the following properties.

Dirichlet bound

$$|\xi - p_n/q_n| \le 1/q_n^2$$

Best approximations

If $0 < q < q_n$ then $|q\xi - p| > |q_n\xi - p_n|$; i.e. the continued fraction convergents are the best rational approximations to ξ.

We say ξ is *badly approximable* if there exists $C' > 0$ such that

$$|\xi - p/q| \ge C'/q^2 \text{ for all } p, q \in \mathbb{Z},$$

i.e. the Dirichlet bound is tight (up to a multiplicative constant). This is an interesting class of real numbers (uncountable, measure zero, Hausdorff dimension 1, etc.) which is not completely understood.

One characterization of badly approximable numbers is the following.

The number $\xi = [a_0; a_1, a_2, \ldots] \in \mathbb{R} \setminus \mathbb{Q}$ is badly approximable if and only its partial quotients a_n are bounded.

[If ξ is badly approximable, $|\xi - p/q| \geq C'/q^2$, then in particular

$$\frac{C'}{q_n^2} \leq |\xi - p_n/q_n| = \frac{1}{q_n^2([a_{n+1}; a_{n+2}, \ldots] + [0; a_n, \ldots, a_1])} \leq \frac{1}{q_n^2 a_{n+1}},$$

$$a_{n+1} \leq 1/C'.$$

Conversely, if the partial quotients are bounded, $\sup_n\{a_n\} \leq M$, then for any p/q with $0 < q \leq q_n$

$$|\xi - p/q| \geq |\xi - p_n/q_n| = \frac{1}{q_n^2(q_{n+1}/q_n + \xi_{n+1})}$$

$$= \frac{1}{q_n^2([0; a_{n+2,\ldots}] + [a_{n+1}; a_n, \ldots, a_1])} \geq \frac{1}{q_n^2\,(a_{n+1} + 2)} \geq \frac{1}{q_n^2(M+2)},$$

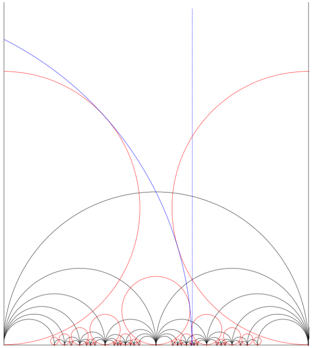
using the fact that the convergents p_n/q_n are the best approximations.]

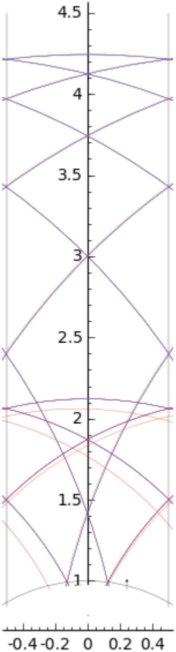
Dani correspondence

The number ξ is badly approximable if and only if the trajectory

$$\Omega_\xi = \left\{ SL_2(\mathbb{Z}) \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$$

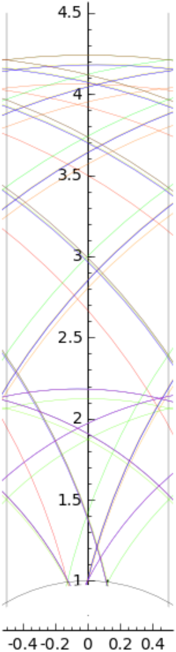
is bounded (precompact).





Left is the trajectory aimed at $\xi = 3\sqrt{2} - 4 = [0; \overline{4, 8}]$. The trajectory is bounded, asymptotic to the closed geodesic joining the conjugate points ξ, $\bar{\xi}$.

Right is the trajectory aimed at the transcendental $\xi = [0; 4, 8, 8, 4, 8, 4, 4, 8, \ldots]$ (digits given by the Thue-Morse sequence on $\{4, 8\}$).



Quadratic irrationals are badly approximable (!!!)

Here are three proofs that real quadratic irrationals

$$Q(x,y) = ax^2 + bxy + cy^2 = 0,\ \ Q(\xi,1) = 0,$$

are badly approximable.

- Partial quotients are eventually periodic: If

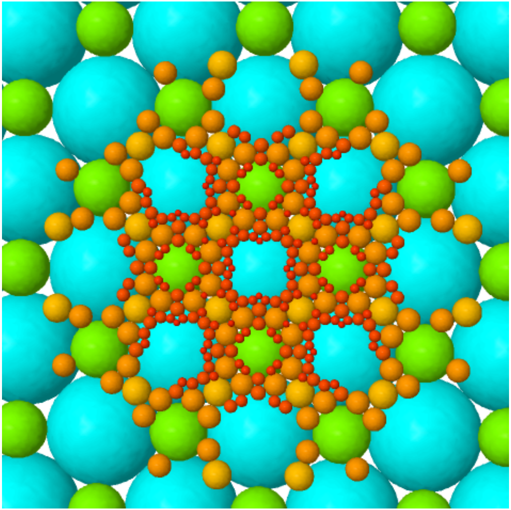
  $$Q_n = Q^{g_n},\ \ g_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},\ \ Q_n(1,\xi_n) = 0,$$

  then $\{Q_n\}_n$ is finite (Dirichlet bound and discreteness of $\mathbb{Z}$).
- Dani correspondence: The geodesic $\overrightarrow{\infty\xi}$ has bounded forward orbit modulo $SL_2(\mathbb{Z})$ since it is asymptotic to a closed geodesic (the projection of $\overrightarrow{\xi'\xi}$).
- Liouville: We have

  $$0 < \frac{1}{q^2} \leq |Q(p/q,1)| = |Q(\xi,1) - Q(p/q,1)| \leq C|\xi - p/q|$$

  since Q is anisotropic and by the mean value theorem.

Moving up a dimension, let's consider approximation to $z \in \mathbb{C}$ over the Euclidean imaginary quadratic fields.



Let $K = \mathbb{Q}(\sqrt{d})$, $d = -1, -2, -3, -7, -11$, an imaginary quadratic field whose maximal order is Euclidean, and let be V the complex numbers closer to zero than to any other point of $\mathcal{O}_K$ along with a choice of half the boundary. Any $z \in \mathbb{C}$ can be uniquely written as

$$z = a_0 + z_0,\ a_0 =: \lceil z \rfloor \in \mathcal{O}_K,\ z_0 =: \{z\} \in V.$$

Define $T : V \to V$ by $Tz = \{1/z\}$. For $z \in \mathbb{C}$, define

$$z_n = T^n z_0 = \{1/z_{n-1}\},\ a_n = \lceil 1/z_{n-1} \rfloor,$$

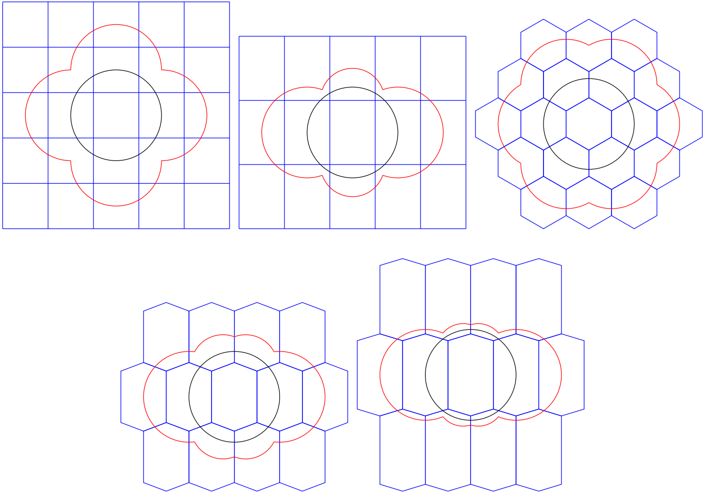
expressing z as a continued fraction

$$z = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

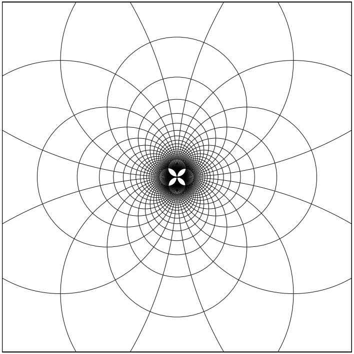
with convergents $p_n/q_n = a_0 + \frac{1}{a_1 + \frac{1}{\ldots + \frac{1}{a_n}}} \in K$

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

Nearest integer continued fractions: Definitions 2



V and its translates in blue, unit circle in black, and $\partial(V^{-1})$ in red.
[Also a “proof by picture” that $\mathcal{O}_K$ is norm Euclidean.]



Partition of V induced by one iteration of T.

The Hurwitz continued fractions aren't as nice as simple continued fractions, for various reasons.

One reason is that the branches of the inverse aren't surjective near the boundary of V, making the sequence space $[a_0; a_1, a_2, \ldots]$ hard to describe (e.g. there are arbitrarily long "forbidden sequences" of digits in nearest integer continued fractions).

Another is that they don't always give the best rational approximations relative to the norm of the denominator.

While the nearest integer convergents p_n/q_n to $z \in \mathbb{C}$ are not necessarily the best rational approximations to z, they aren't so bad.

Dirichlet bound

There exists $C > 0$ such that the convergents p_n/q_n to any $z \in \mathbb{C}$ satisfy

$$|z - p_n/q_n| \leq C/|q_n|^2.$$

OK approximations

There exists $\alpha > 0$ such that for any $z \in \mathbb{C}$, $p, q \in \mathcal{O}_K$, $|q| \leq |q_n|$,

$$\alpha|qz - p| \geq |q_n z - p_n|.$$

[The second statement is essentially due to R. Lakein, who found

$$\sup_{z,n} |q_n(q_n z - p_n)|$$

for the algorithms considered here. It can also be found implicitly in the work of D. Hensley and explicitly S. G. Dani over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively.]

Monotonicity of q_n

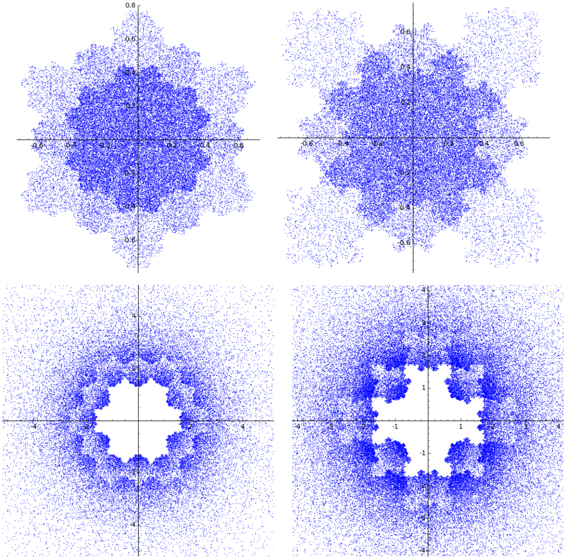
Need to establish

For the algorithms above, the convergent denominators are increasing in norm:

$$|q_n| < |q_{n+1}|.$$

for the results of the last slide. The fractals indicate the difficulty of describing the natural extension and the problem with small partial quotients, since

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_1].$$



While the nearest integer convergents are not the best approximations available, they are “good enough” to detect badly approximable numbers.

The number $z \in \mathbb{C} \setminus K$ is badly approximable over K if and only if its partial quotients are bounded.

[The proof follows from the approximation properties described earlier, with the “OK approximations” allowing us to say that a number is badly approximable if and only if it is badly approximable by its convergents. This statement for $\mathbb{Q}(\sqrt{-3})$ can also be found in a recent preprint of S. G. Dani.]

The zero set $Z(H)$ of the indefinite binary Hermitian form

$$H(z,w) = (\bar{z}\ \bar{w}) \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = A|z|^2 + B\bar{z}w + \overline{B}z\bar{w} + C|w|^2,$$

$$A, C \in \mathbb{R}, B \in \mathbb{C},\ \Delta(H) := \det(H) < 0,$$

is a circle in $P^1(\mathbb{C})$; e.g. if $A \neq 0$ then

$$Z(H) \cap \mathbb{C}_z = \{z : |z + B/A|^2 = -\Delta/A^2\}.$$

$GL_2(\mathbb{C})$ acts on a form H by change of variable and on the circle $Z(H)$ by the usual Möbius action, and the map $H \to Z(H)$ is $GL_2(\mathbb{C})$-equivariant:

$$Z(g^\dagger H g) = g^{-1} \cdot Z(H),\ g \in GL_2(\mathbb{C}).$$

A form/circle is *rational* if $A, B, C \in K$ and *integral* if $A, B, C \in \mathcal{O}_K$. We can restrict the actions above to $GL_2(\mathcal{O}_K)$ and integral forms.

For $z \in \mathbb{C}$ with $z = [a_0; a_1, \ldots]$, define

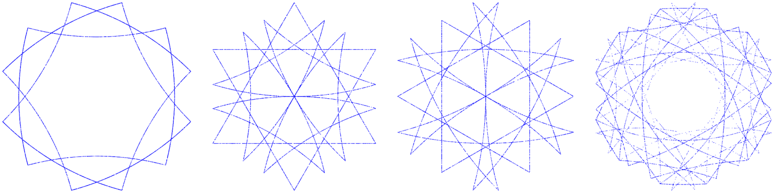
$$g_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

so that $T^n(z_0) = 1/g_n^{-1}z$. If $H(z,1) = 0$ for an indefinite integral binary Hermitian form, let $H_n = H^{g_n}$, so that $H_n(1, z_n) = 0$.

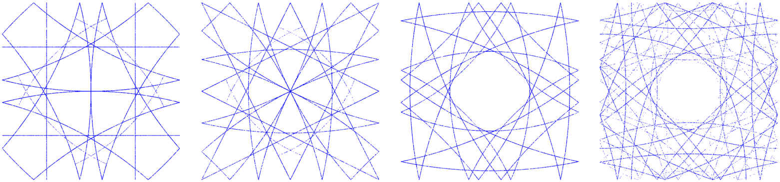
The orbit $\{H_n\}_n$ is finite. [Hence the entire orbit $\{T^n(z_0) : n \geq 0\}$ lies on finitely many circles.]

This is analogous to the fact that real irrational quadratic numbers have eventually periodic simple continued fraction expansions as discussed earlier.

However, the sequences H_n and z_n aren't periodic (unless z is a quadratic irrational); there is room to move around on the tagged circles $(z_n, Z(H_n))$. [This generalizes work of Wieb Bosma and David Gruenewald over $\mathbb{Q}(\sqrt{-1})$.]



Example orbits $\{z_n\}_n$, $0 \leq n \leq 10,000$ for zeros z of various integral forms over $\mathbb{Q}(\sqrt{-3})$.



Example orbits $\{z_n\}_n$, $0 \leq n \leq 20,000$ for zeros z of various integral forms over $\mathbb{Q}(\sqrt{-1})$.

[Now K is *any* imaginary quadratic field.]

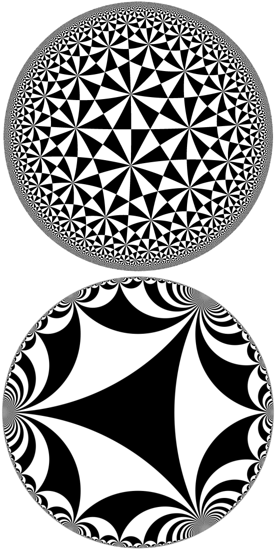
A circle $Z(H)$ in the plane determines a geodesic plane (hemisphere) $S(H)$ in $\mathcal{H}^3$ (upper half-space model). In the quotient $\pi : \mathcal{H}^3 \to SL_2(\mathcal{O}_K)\backslash\mathcal{H}^3$ we get some geodesic surface $\pi(S(H))$.

If H is an anisotropic rational form, then $\pi(S(H))$ is compact. [Equivalently $SU(H, \mathcal{O}_K)\backslash SU(H, \mathbb{C})$ is compact.]

H is *anisotropic* if $H(z, w) \neq 0$ for $[z : w] \in P^1(K)$. This is equivalent to the condition

$$-\Delta(H) \notin N_{\mathbb{Q}}^K(K),$$

i.e. the square of the radius of the rational circle is not a norm.



Dani correspondence

The number z is badly approximable if and only if the trajectory

$$\Omega_z = \left\{ SL_2(\mathcal{O}_K) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : t \geq 0 \right\} \subseteq SL_2(\mathcal{O}_K) \backslash SL_2(\mathbb{C})$$

is bounded (precompact).

If $z \in \mathbb{C}$ satisies $H(z, 1) = 0$ for some anisotropic integral form H, then z is badly approximable.

- If K is Euclidean, then $\{Z(H_n)\}_n$ is a finite collection of circles bounded away from zero/infinity so that the partial quotients of z are bounded (and all approximation constants are effective).
- For general K, the trajectory Ω_z is asymptotic to the compact geodesic surface $\pi(S(H))$, and is therefore bounded.

[The collection of badly approximable points produced is uncountable of measure zero, dense in the plane, and of Hausdorff dimension 1. The collection of all numbers badly approximable over K has Hausdorff dimension 2.]

On these anisotropic circles, there are many examples of algebraic numbers badly approximable over K.

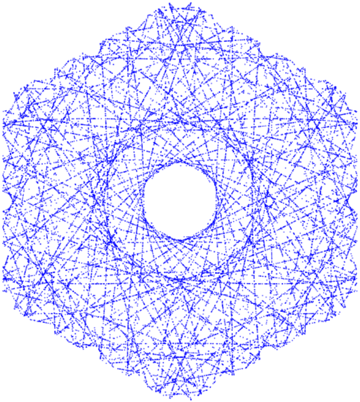
For any real algebraic number $u \in [-2, 2]$, any $0 < n \in \mathbb{Q} \setminus N_{\mathbb{Q}}^{K}(K)$, and any $t \in K$, the number

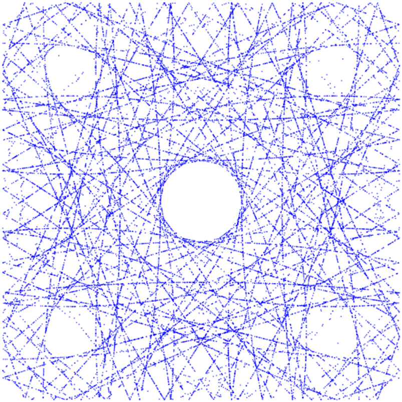
$$z = t + \sqrt{n} \cdot \frac{u \pm \sqrt{u^2 - 4}}{2}$$

is badly approximable. Moreover, this parameterizes all of the algebraic numbers badly approximable over K coming from rational circles.

[Examples of algebraic numbers with bounded partial quotients over $\mathbb{Q}(\sqrt{-1})$ were given by Bosma and Gruenewald generalizing examples of Hensley.]

For instance, 30,000 iterates of T on the quadratically scaled root of unity $z = \sqrt{23}e^{2\pi i/5}$ over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$ ($23 \notin N_{\mathbb{Q}}^{K}$):





Suppose H is an anisotropic indefinite integral binary Hermitian form over an imaginary quadratic field with $H(z,1)=0$. Then Liouville-style estimates

$$0 < 1/|q|^2 \leq |H(p/q,1)| = |H(p/q,1) - H(z,1)| \leq C|z - p/q|,$$

show the following.

With H and z as above, we have

$$\liminf_{|q|\to\infty} \{|q(qz-p)| : p,q \in \mathcal{O}, q \neq 0\} \geq \frac{\mu}{2\sqrt{-\Delta}},$$

where $\mu = \min\{|H(p,q)| : (0,0) \neq (p,q) \in \mathcal{O}_K^2\}$ and $\Delta = \det(H)$.

We can generalize the quadratic/Hermitian examples of badly approximable numbers above to vectors in $F \otimes \mathbb{R}$.

- F a number field of signature (r, s)
- When pertinent, F/E CM (F/E imaginary quadratic, E totally real). The importance here is that "complex conjugation" needs to commute with other automorphisms.
- $G = SL_2(F \otimes \mathbb{R})$, $\Gamma = SL_2(\mathcal{O}_F)$, $K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s$ (maximal compact), $G/K \cong (\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$
- $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$

We will measure the approximation of $\mathbf{z} = (z_\sigma)_\sigma$ by $p/q \in F$ with

$$\max_\sigma\{|\sigma(q)|\} \max_\sigma\{|\sigma(q) z_\sigma - \sigma(p)|\},$$

the product of the sizes of the (vector-valued) linear forms q and $q\mathbf{z} - p$.

We say $\mathbf{z}$ is *badly approximable* if there exists $C' > 0$ such that

$$\max_\sigma\{|\sigma(q)|\} \max_\sigma\{|\sigma(q) z_\sigma - \sigma(p)|\} \geq C'$$

for all $p/q \in F$.

[If $\mathbf{z}$ is badly approximable as defined above, then

$$\max_\sigma\{|z_\sigma - \sigma(p/q)|\} \geq C' / \max_\sigma\{|\sigma(q)|\}^2,$$

and the converse holds for $(r, s) = (1, 0)$, $(0, 1)$, $(2, 0)$, $(0, 2)$.]

The measure of approximation introduced above has the following properties.

Dirichlet-type

There exists $C > 0$ such that any $\mathbf{z} \notin F$ has infinitely many rational approximations p/q with

$$\max_{\sigma}\{|\sigma(q)|\} \max_{\sigma}\{|\sigma(q)z_{\sigma} - \sigma(p)|\} \leq C.$$

Roth-type

For any algebraic $\mathbf{z} \notin F$ (each z_{σ} algebraic) and any $\epsilon > 0$, there exists $C' > 0$ such that

$$\max_{\sigma}\{|\sigma(q)|\}^{1+\epsilon} \max_{\sigma}\{|\sigma(q)z_{\sigma} - \sigma(p)|\} \geq C'$$

for any $p/q \in F$.

The set of badly approximable vectors has measure zero but full Hausdorff dimension.

Some previous work

*M. Einsiedeler, A. Ghosh, and B. Lyttle: The set of badly approximable vectors is "winning" (even when restricted to curves) in the setting above.
*D. Kleinbock and T. Ly: The set of badly approximable vectors is "$\mathcal{H}$-absolute winning" (even when restricted to curves and some fractals) in the setting above.
*T. Hattori: Proved Dirichlet-type theorems for real quadratic and complex quartic fields - infinitely many solutions to

$$\|\mathbf{z} - p/q\|_1 \leq C/\sqrt{H(q)},$$

and gave examples of badly approximable vectors.
*R. Quême: Proved Dirichlet-type theorems - infinitely many solutions to

$$\|q\|_{1'} \cdot \|q\mathbf{z} - p\|_{1'} \leq C, \;\; N(\mathbf{z} - p/q) \leq C/N(q)$$

where $N(\mathbf{z})$ is the extension of the absolute value of the field norm.
*E. Burger: Diophantine approximation over S-integers and examples of badly approximable linear systems using

$$h_S(\mathbf{x}, \mathbf{y})^N \prod_{v \in S} |A_v \mathbf{x} - \mathbf{y}|_v^M, \;\; \mathbf{x} \in \mathcal{O}_F^N, \;\; , \mathbf{y} \in \mathcal{O}_F^M, \;\; A_v \in Mat_{M \times N}(F_v).$$

*W. Schmidt: Dirichlet-type theorem (for $\prod_\sigma \sigma(K) \subseteq \mathbb{R}^r \times \mathbb{C}^s$) and the Subspace theorem for number fields (from which the above Roth-type theorem can be deduced).

*S. G. Dani: Characterization of badly approximable systems of linear forms in terms of bounded trajectories in $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$.

Dani correspondence

The vector $\mathbf{z}$ is badly approximable over F if and only if the trajectory

$$\omega_{\mathbf{z}}(t) = \left\{ \Gamma \cdot \left(\begin{pmatrix} 1 & z_\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{\sigma \in S} \cdot K : t \geq 0 \right\}$$

is bounded in $\Gamma \backslash G / K$

This follows from Mahler's compactness criterion (tailored here to our current needs):

Mahler's criterion

A subset $\Omega \subseteq SL_2(F \otimes \mathbb{R})$ is precompact modulo Γ if and only if the $\mathcal{O}_F$-lattices spanned by the rows of elements of $\omega \in \Omega$ have no arbitrarily short vectors, i.e. there exists $\epsilon > 0$ such that

$$\inf\{\|(p\ q)\omega\| : (0,0) \neq (p,q) \in \mathcal{O}_F^2,\ \omega \in \Omega\} \geq \epsilon.$$

Let J be an F-rational binary quadratic or Hermitian form (need F/E to be CM to define Hermitian forms). Let J_σ be the form obtained by applying σ to the coefficients of J.

We say J is *totally indefinite* if J_σ is indefinite for all σ. For such J, we have its zero set

$$Z(J) = \prod_\sigma Z(J_\sigma) \subseteq (P^1(\mathbb{R}))^r \times (P^1(\mathbb{C}))^s$$

and $g \in SL_2(\mathcal{O}_F)$ acts by change of variable on J or by fractional linear transformations on $Z(J)$ (equivariantly).

If J is anisotropic (no zeros in $P^1(F)$), then

$$(\mathrm{Aut}(J) \cap \Gamma)\backslash \mathrm{Aut}(J) \subseteq \Gamma\backslash G$$

is compact (follows from Mahler's criterion), and this compact set is (almost) the product of lines/planes associated to $Z(J)$.

Applying the Dani correspondence to geodesic trajectories aimed at $Z(J)$ (asymptotic to the compact subspaces just described), we get:

Let J be totally indefinite anisotropic F-rational binary quadratic or Hermitian form and $\mathbf{z} \in Z(J) \subseteq \mathbb{R}^r \times \mathbb{C}^s$. Then $\mathbf{z}$ is badly approximable.

[Once again there is also an elementary, Liouville-style proof.]

Example 1: $Q = x^2 - (2-\sqrt{2})y^2$ is anisotropic and totally indefinite over $\mathbb{Q}(\sqrt{2})$, so the four vectors

$$\left(\pm\sqrt{2-\sqrt{2}}, \pm\sqrt{2+\sqrt{2}}\right) \in \mathbb{R}^2$$

are badly approximable.

Example 2: $H = |z|^2 - 3|w|^2$ is anisotropic and totally indefinite over $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$, so every vector in the torus

$$\{(\sqrt{3}\cos s + i\sqrt{3}\sin s, \sqrt{3}\cos t + i\sqrt{3}\sin t) : s, t \in [0, 2\pi)\} \subseteq \mathbb{C}^2$$

is badly approximable.

We would like to note that the $Z(H)$ contain many algebraic vectors, which we can parameterize as follows.

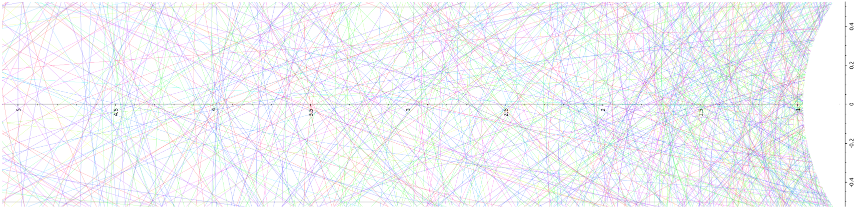
Choose real algebraic numbers $u_\sigma \in [-2, 2]$, a totally positive $t \in E \backslash N_E^F(F)$, and any $s \in F$. Then

$$\mathbf{z} = (z_\sigma)_\sigma, \ z_\sigma = \sigma(s) + \sqrt{\sigma(t)} \, \frac{u_\sigma \pm \sqrt{u_\sigma^2 - 4}}{2},$$

are the algebraic badly approximable vectors associated to Hermitian forms.

Back to $\mathbb{Q} \subseteq \mathbb{R}$

Let's take another look at continued fractions, with a focus on reflection groups and some of the ergodic theory.



[A slice of π.]

Reflective Continued Fractions

Consider the group $\Gamma = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \rangle \subseteq PGL_2(\mathbb{R}) \cong \mathrm{Isom}(\mathcal{H}^2)$, generated by reflections in the walls of the ideal hyperbolic triangle with vertices $\{0, 1, \infty\}$:

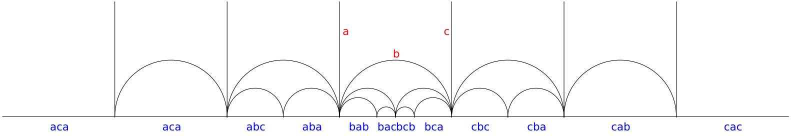
$$\mathfrak{a}(x) = -x, \ \mathfrak{b}(x) = \frac{x}{2x-1}, \ \mathfrak{c}(x) = 2 - x.$$

We have

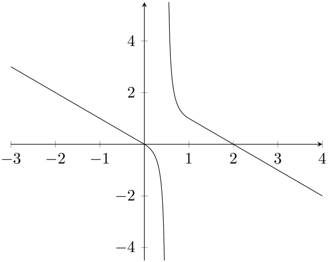
$$1 \to \Gamma \to PGL_2(\mathbb{Z}) \to PGL_2(\mathbb{Z}/(2)) \to 1,$$

$$PGL_2(\mathbb{Z}) = \Gamma \rtimes S_3, \ \Gamma \cong \mathbb{Z}/(2) * \mathbb{Z}/(2) * \mathbb{Z}/(2).$$

Words in these generators index the triangles in the tessellation, and the words of length n partition the line into $3 \cdot 2^{n-1}$ subintervals. The partition by words of length $m > n$ refines the partition by words of length n. Irrational x are then uniquely coordinatized by infinite words in the alphabet $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$.



$$T(x) = \begin{cases} \mathfrak{a}(x) = -x & x \in [-\infty, 0] \\ \mathfrak{b}(x) = \frac{x}{2x-1} & x \in [0, 1] \\ \mathfrak{c}(x) = 2 - x & x \in [1, \infty] \end{cases}$$



The expansion of x as an infinite word in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is produced by a dynamical system $T : P^1(\mathbb{R}) \to P^1(\mathbb{R})$.

There are three neutral fixed points, $\frac{0}{1}$, $\frac{1}{1}$, and $\frac{1}{0}$, to which rational points descend in finitely many steps depending on the parity of the numerator and denominator (even/odd, odd/odd, odd/even).

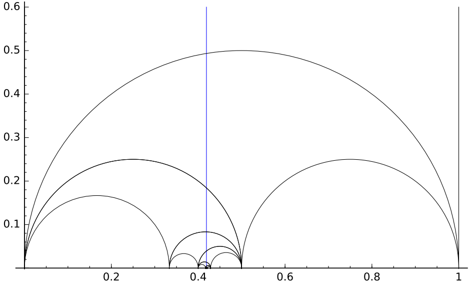
If x is irrational and $\mathfrak{m} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ is defined by $T^n x = \mathfrak{m}(T^{n-1}x)$, then we have three sequences of rational convergents

$$\lim_{n\to\infty} \mathfrak{m}_1\mathfrak{m}_2 \ldots \mathfrak{m}_n x_0 = x, \ \ x_0 = 0, 1, \infty,$$

the vertices of the triangles through which the geodesic $\overrightarrow{\infty x}$ passes.

These approximations can also be constructed as a sequence of mediants starting with $(\frac{1}{1}, \frac{1}{0}, \frac{0}{1})$ or $(\frac{1}{1}, \frac{-1}{0}, \frac{0}{1})$.

Example



Here is a random number

$x = 0.4189513796210592\ldots$

with expansion

$\mathfrak{x} = \mathfrak{bacabcacbcacacababac}\ldots$

The first 20 convergents are ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ updating positions 1, 2, 3, mediant in red)

$(1/1, 1/0, 0/1)$:
$(1/1, 1/2, 0/1)$, $(1/3, 1/2, 0/1)$, $(1/3, 1/2, 2/5)$,
$(3/7, 1/2, 2/5)$, $(3/7, 5/12, 2/5)$, $(3/7, 5/12, 8/19)$,
$(13/31, 5/12, 8/19)$, $(13/31, 5/12, 18/43)$, $(13/31, 31/74, 18/43)$,
$(13/31, 31/74, 44/105)$, $(75/179, 31/74, 44/105)$, $(75/179, 31/74, 106/253)$,
$(137/327, 31/74, 106/253)$, $(137/327, 31/74, 168/401)$, $(199/475, 31/74, 168/401)$,
$(199/475, 367/876, 168/401)$, $(535/1277, 367/876, 168/401)$, $(535/1277, 703/1678, 168/401)$,
$(871/2079, 703/1678, 168/401)$, $(871/2079, 703/1678, 1574/3757)$
$= (0.4189514\ldots, 0.4189511\ldots, 0.4189512\ldots)$.

The invertible extension $\widetilde{T}$ of T is defined on the space of geodesics $\mathcal{G}$ that intersect the fundamental triangle, acting on $\overrightarrow{yx}$ depending on x:

$$\widetilde{T}(y,x) = \begin{cases} (\mathfrak{a}(y), \mathfrak{a}(x)) & x \in [-\infty, 0] \\ (\mathfrak{b}(y), \mathfrak{b}(x)) & x \in [0, 1] \\ (\mathfrak{c}(y), \mathfrak{c}(x)) & x \in [1, \infty] \end{cases}$$

$\widetilde{T}$ associates to the geodesic $\overrightarrow{yx}$ a bi-infinite word $\mathfrak{y}^{-1}\mathfrak{x}$ in $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, which we will relate to the geodesic flow in $\Gamma \backslash \mathcal{H}^2$.

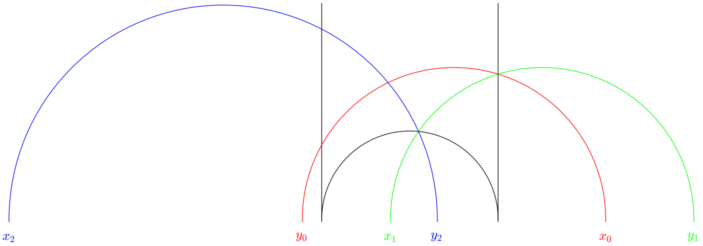
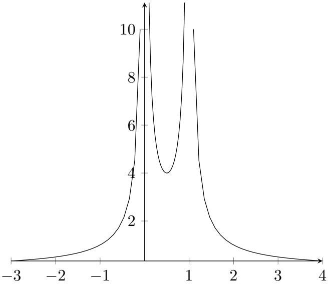


Figure: Two iterations of $\widetilde{T}$, red to green to blue

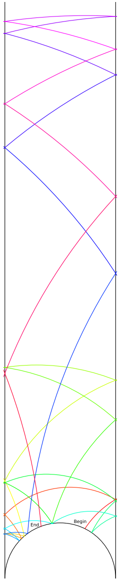
$$d\mu(x) = \begin{cases} \frac{dx}{-x} & x < 0, \\ \frac{dx}{x(1-x)} & 0 < x < 1, \\ \frac{dx}{x-1} & x > 1, \end{cases}$$



The measure $d\eta(y,x) = (x-y)^{-2}dxdy$ is isometry-invariant on the space of geodesics in the hyperbolic plane.

Since $\widetilde{T}$ is a bijection defined piecewise by isometries, $\eta|_{\mathcal{G}}$ is $\widetilde{T}$-invariant.

Pushing forward to the second coordinate gives an **infinite** T-invariant measure μ. We will see that $(\mathcal{G}, \widetilde{T}, \eta|_{\mathcal{G}})$ is ergodic (hence also the ergodicity of $(P^1(\mathbb{R}), T, \mu)$).



The word $\mathfrak{y}^{-1}\mathfrak{x}$ associated to the geodesic $\overrightarrow{yx}$ records the sequence of collisions with the walls of the triangle in $\Gamma\backslash\mathcal{H}^2$, and $\widetilde{T}$ is the first-return of the geodesic flow (billiards in the triangle) to the cross-section defined by points/directions on the boundary. The return time is integrable with respect to $d\eta(y,x)$.

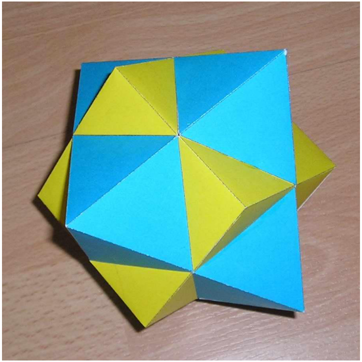
[For instance, a geodesic $(y,x) \in [-\infty, 0] \times [1, \infty]$, has retrun time

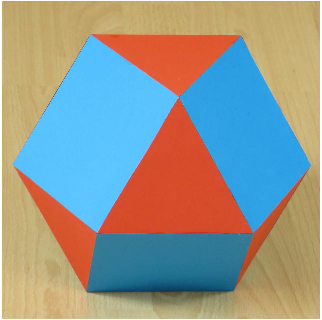
$$r(y,x) = \frac{1}{2}\log\left(\frac{x(1-y)}{y(1-x)}\right).$$

and the integral is

$$\frac{1}{2}\int_{-\infty}^{0}\int_{1}^{\infty}\log\left(\frac{x(1-y)}{y(1-x)}\right)\frac{dxdy}{(y-x)^2} = \frac{\pi^2}{6}.]$$

I said all that to say this: Right-angled continued fractions



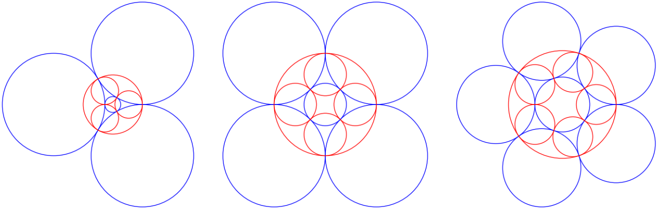


Let Π be a polyhedron whose faces can be two-colored, i.e. the faces of Π can be partitioned into two sets $S = \{s_i\}$ and $T = \{t_j\}$ such that no two faces in S share an edge and no two faces in T share an edge. We consider right-angled hyperbolic Coxeter groups of the form

$$\Gamma = \langle s_1, \ldots, s_m, t_1, \ldots, t_n | s_i^2 = t_i^2 = [s_i, t_j] = 1, s_i \sim t_j \rangle,$$

where $s_i \sim t_j$ if the faces s_i and t_j share an edge.

The ideal boundaries of the planes defining the faces of Π consist of oriented circles $Z(s_i)$, $Z(t_j)$ with the property that the interiors of $Z(s)$, $s \in S$ are disjoint, the interiors of $Z(t)$, $t \in T$ are disjoint, together they cover the sphere, and if $Z(s)$ and $Z(t)$ intersect, they do so at right angles.



Define two dynamical systems on $S^2 = P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ as follows

$$\phi_S(z) = \begin{cases} s(z) & z \in C(s) \\ t(z) & z \in P(t) \end{cases}, \quad \phi_T(w) = \begin{cases} s(w) & w \in P(s) \\ t(w) & w \in C(t) \end{cases},$$

where P and C are the polygonal (interstitial) and circular regions associated to the Coxeter generator. The invertible extensions Φ_S and Φ_T are defined on a collection $\mathcal{G}$ of geodesics (pairs of distinct points of $P^1(\mathbb{C})$):

$$\Phi_S(w,z) = (r(w), \phi_S(z)) \;\text{ where }\; \phi_S(z) = r(z),\; r \in S \cup T,$$

$$\Phi_T(w,z) = (\phi_T(w), q(z)) \;\text{ where }\; \phi_T(w) = q(w),\; q \in S \cup T.$$

Moreover (surprisingly?), the extensions Φ_S and Φ_T are inverse to one another.

The maps ϕ and Φ are one- and two-sided subshifts of finite type on the alphabet $S \cup T$. The sequences in question come from the two "obvious" normal forms for elements of Γ. If we consider ϕ_S, then the sequence

$$(\phi_S(z), \phi_S^2(z), \phi_S^3(z), \dots) = (r_1(z), r_2 r_1(z), r_3 r_2 r_1(z), \dots), \; r_n \in S \cup T,$$

has the following properties.

- $r_n \neq r_{n+1}$, i.e. no words of the form s^2 or t^2 appear (inversions in the boundary circles of $C(s)$ or $P(t)$ ensure you do not repeat a digit).
- If r_n and r_{n+1} commute (i.e. their fixed circles are orthogonal), then $r_n \in S$ and $r_{n+1} \in T$. Geometrically, this comes from the fact that we are taking $C(s)$ and $P(t)$ in the definition, i.e. we prefer S over T in the definition.

Similarly, the map ϕ_T will produce a sequence

$$(\phi_T(w), \phi_T^2(w), \phi_T^3(w), \dots) = (q_1(w), q_2 q_1(w), q_3 q_2 q_1(w), \dots), \; q_n \in S \cup T,$$

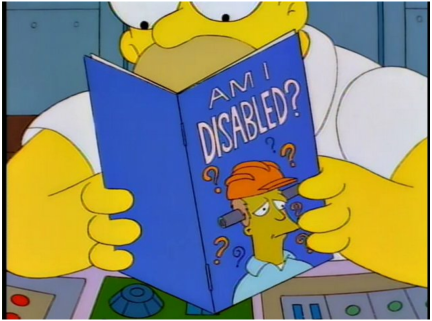
satisfying the opposite convention for the commutation relations: if q_n and q_{n+1} commute, then $q_n \in T$ and $q_{n+1} \in S$.
Hence, to a point $w \in \mathbb{C}$ or $z \in \mathbb{C}$, we encode the pair (w, z) as

$$(\dots q_3 q_2 q_1, r_1 r_2 r_3 \dots).$$

We can explicitly compute invariant measures for ϕ_S and ϕ_T by integrating the isometry invariant measure

$$|z-w|^{-4}dxdydudv, \ z=x+iy, \ w=u+iv, \ (w,z)\in\mathcal{G},$$

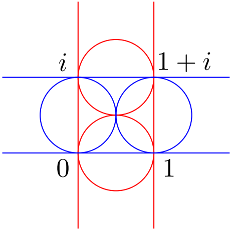
over w or z.



Unfortunately, I haven't been able to show that these measure preserving systems are ergodic.

How 'bout some (interesting) examples? [These are variations on algorithms of A. L. Schmidt.]

We will work with the following group of Möbius transformations, reflections in the sides of the (finite volume) ideal right-angled octahedron with vertices $\{0, 1, \infty, i, 1+i, 1/(1-i)\}$



$$\Gamma = \langle \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_1^\perp, \mathfrak{s}_2^\perp, \mathfrak{s}_3^\perp, \mathfrak{s}_4^\perp \rangle \subseteq PSL_2(\mathbb{C}) \rtimes \langle \bar{z} \rangle \cong \mathrm{Isom}(H^3),$$

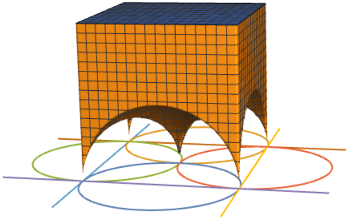
$$\mathfrak{s}_1 = \frac{(1+2i)\bar{z} - 2}{2\bar{z} - 1 + 2i}, \ \mathfrak{s}_2 = \frac{\bar{z}}{2\bar{z} - 1}, \ \mathfrak{s}_3 = -\bar{z} + 2, \ \mathfrak{s}_4 = -\bar{z},$$

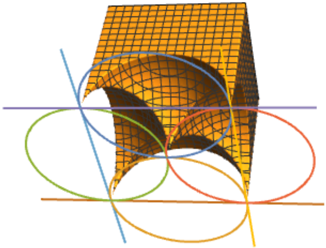
$$\mathfrak{s}_1^\perp = \bar{z}, \ \mathfrak{s}_2^\perp = \bar{z} + 2i, \ \mathfrak{s}_3^\perp = \frac{\bar{z}}{-2i\bar{z} + 1}, \ \mathfrak{s}_4^\perp = \frac{(1-2i)\bar{z} + 2i}{-2i\bar{z} + 1 + 2i},$$

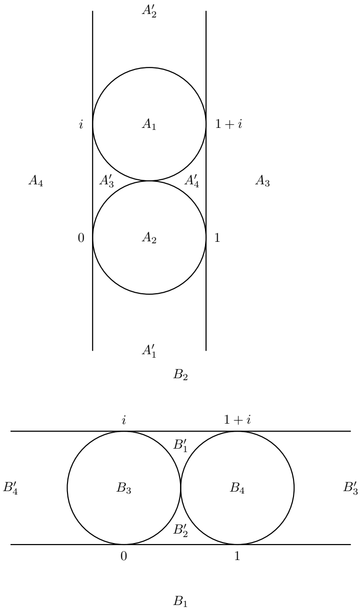
$$1 \to \Gamma \to PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle \to PGL_2(\mathbb{Z}[i]/(2)) \to 1,$$

$$[PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle : \Gamma] = 48, \ PGL_2(\mathbb{Z}[i]) \rtimes \langle \bar{z} \rangle = \Gamma \rtimes \text{Bin. Oct.}$$

Just in case you didn't see an octahedron:







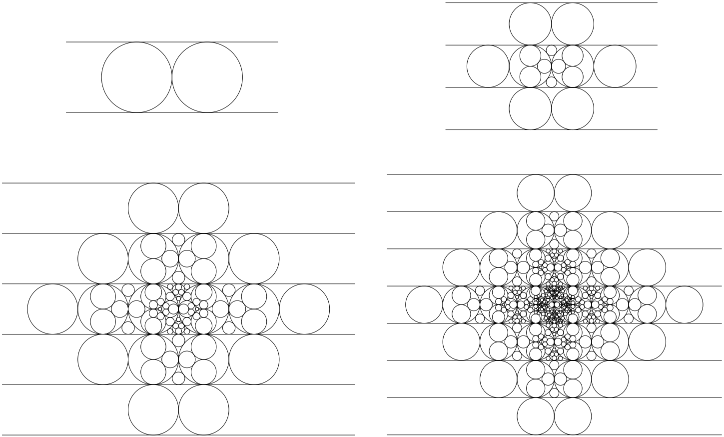
As discussed earlier, we have two dynamical systems:

$$T_A(w) = \begin{cases} \mathfrak{s}_i w & w \in A_i, \\ \mathfrak{s}_i^{\perp} w & w \in A_i'. \end{cases},$$

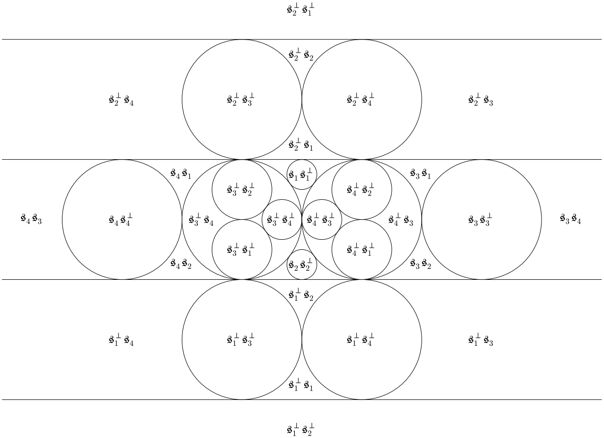
$$T_B(z) = \begin{cases} \mathfrak{s}_i z & z \in B_i', \\ \mathfrak{s}_i^{\perp} z & z \in B_i, \end{cases}.$$

The vertices $\left\{0, 1, \infty, i, 1+i, \frac{1}{1-i}\right\}$ are neutral fixed points to which Gaussian rationals descend in finite time, depending on the "parity" of the numerator and denominator (because Γ is two-congruence).

Sequence of partitions associated to T_B:



The regions of the nth partition are labeled by the $9 \cdot 5^{n-1} - 1$ normal form words of length n. Irrational z are uniquely coordinatized by infinite normal form words in the generators $\{\mathfrak{s}_i, \mathfrak{s}_i^\perp\}$. The expansion of z in the generators is produced by the dynamical system (T_B here).



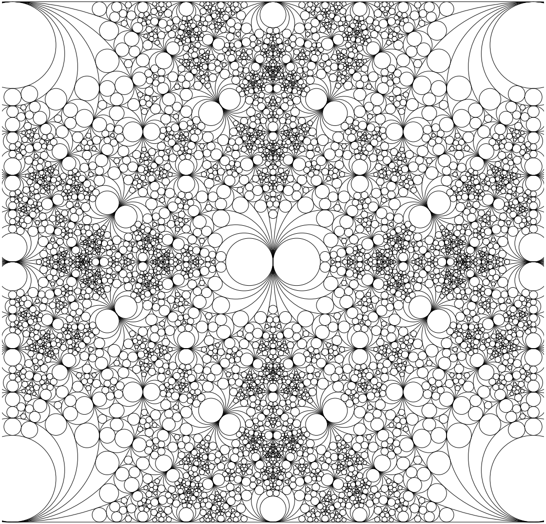


Figure: Portion of the sixth partition inside the unit square.

Recording the sequences $\mathfrak{m}_n$, $\mathfrak{n}_n$ defined by

$$T_A^n(w) = \mathfrak{m}_n(T_A^{n-1}(w)),\ T_B^n(z) = \mathfrak{m}_n(T_B^{n-1}(z))$$

produces infinite words in normal form. The initial segments $\mathfrak{m}_1 \cdots \mathfrak{m}_n$, $\mathfrak{n}_1 \cdots \mathfrak{n}_n$ label the region in the nth partition where w or z lies. We obtain 6 sequences of Gaussian rational approximations for each system such that

$$\begin{array}{c} \lim\limits_{n\to\infty} \mathfrak{m}_1 \cdots \mathfrak{m}_n w_0 = w \\ \lim\limits_{n\to\infty} \mathfrak{n}_1 \cdots \mathfrak{n}_n z_0 = z \end{array} \quad w_0, z_0 \in \left\{0, 1, \infty, i, 1+i, \frac{1}{1-i}\right\},$$

the vertices of the octahedra along the path indexed by the normal form word.

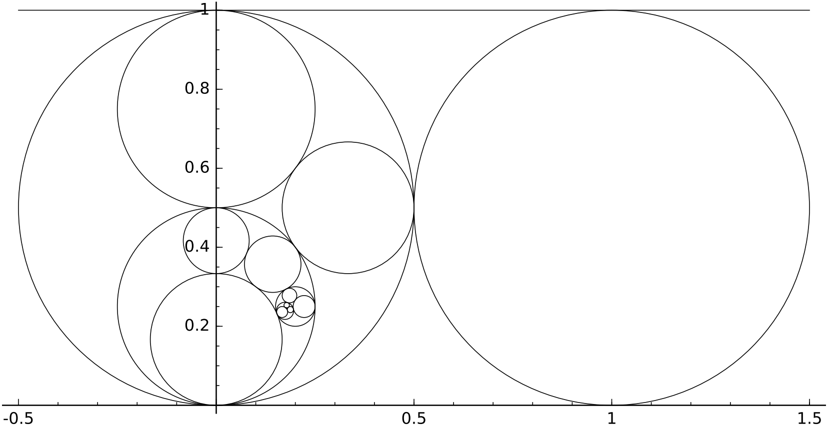
One measure of the quality of approximation is:

If $(p, q) = 1$ is such that

$$|z - p/q| < \frac{C}{|q|^2},\ C = \frac{1}{1 + 1/\sqrt{2}},$$

then p/q is a convergent to z. Moreover, C is the largest constant possible.

Example



$$z = 0.1761148094996705\ldots + i0.2463661645805464\ldots$$

$$\mathfrak{z} = \mathfrak{s}_3^\perp \mathfrak{s}_1^\perp \mathfrak{s}_2 \mathfrak{s}_2^\perp \mathfrak{s}_3^\perp \mathfrak{s}_3 \mathfrak{s}_1 \mathfrak{s}_3 \mathfrak{s}_3^\perp \mathfrak{s}_2^\perp \mathfrak{s}_1 \mathfrak{s}_4 \mathfrak{s}_1 \mathfrak{s}_4 \mathfrak{s}_4^\perp \mathfrak{s}_1 \mathfrak{s}_1^\perp \mathfrak{s}_2^\perp \mathfrak{s}_3 \mathfrak{s}_4 \ldots$$

The invertible extension $\widetilde{T}$ of T_B is defined on a space of geodesics $\mathcal{G} = \cup_i \left(\mathcal{B}_i \times B_i \cup \mathcal{B}'_i \times B'_i\right)$

$$\widetilde{T}(w,z) = \begin{cases} (\mathfrak{s}_i w, \mathfrak{s}_i z) & z \in B_i,\ \mathfrak{z} = \mathfrak{s}_i \ldots \\ (\mathfrak{s}_i^{\perp} w, \mathfrak{s}_i^{\perp} z) & z \in B'_i,\ \mathfrak{z} = \mathfrak{s}_i^{\perp} \ldots \end{cases}$$

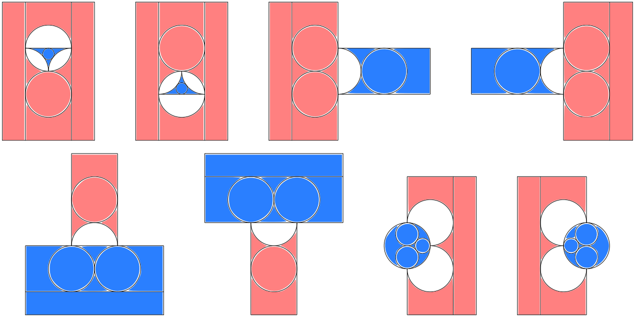
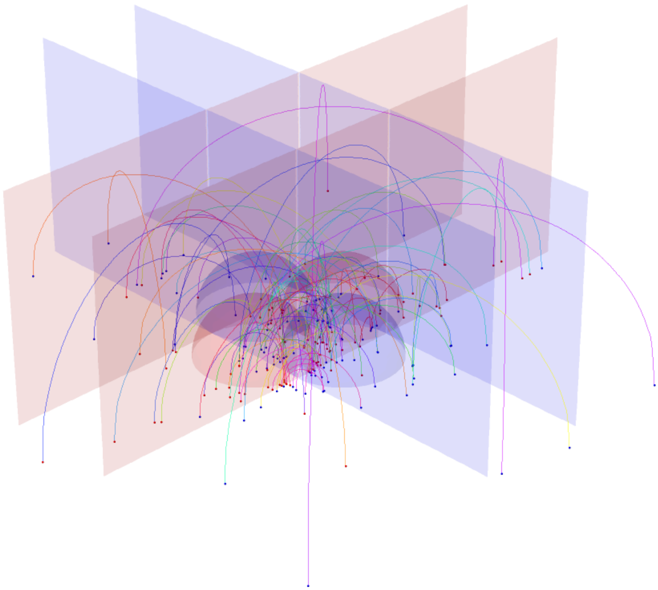
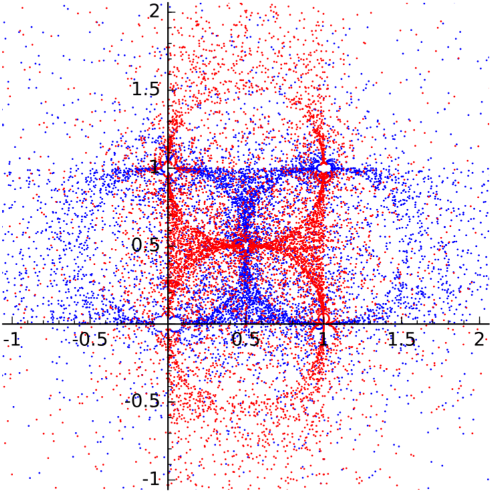


Figure: Regions $\mathcal{B}'_i \times B'_i$, $\mathcal{B}_i \times B_i$ with subdivisions.



An example orbit (100 iterations on random input).



The inverse $\widetilde{T}^{-1}$ extends T_A (in the same manner that $\widetilde{T}$ extends T_B) so that a geodesic (w, z) corresponds to a bi-infinite word $\mathfrak{w}^{-1}\mathfrak{z}$, with $\mathfrak{w}$ produced by T_A and $\mathfrak{z}$ produced by T_B. Hence working with one system automatically involves its dual. [Pictured is part of 10,000 iterations of $\widetilde{T}$ on a random input (w, z)]

The measure

$$d\eta(w,z) = |z-w|^{-4} du dv dx dy,\ \ z = x+iy,\ \ w = u+iv$$

is isometry-invariant on the space of geodesics in H^3. As $\widetilde{T}$ is a bijection defined piecewise by isometries, $\eta|_{\mathcal{G}}$ is $\widetilde{T}$-invariant. Pushing forward to the second coordinate gives a T_B-invariant measure μ_B on $P^1(\mathbb{C})$:

$$d\mu_B(z) = f_B(z) dx dy = \begin{cases} dx dy \int_{\mathcal{B}_i} |z-w|^{-4} du dv, & z \in B_i \\ dx dy \int_{\mathcal{B}'_i} |z-w|^{-4} du dv, & z \in B'_i \end{cases}.$$

[These integrals are explicitly computable.]

The measure μ_B is **finite**, giving a measure of $\pi^2/4$ for each of the eight regions.

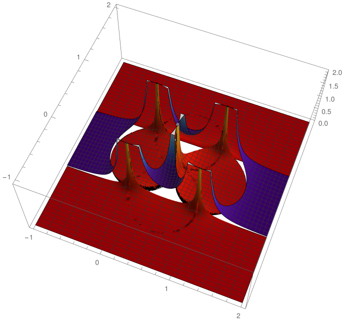
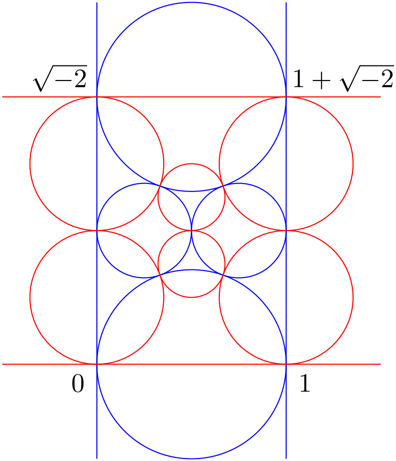
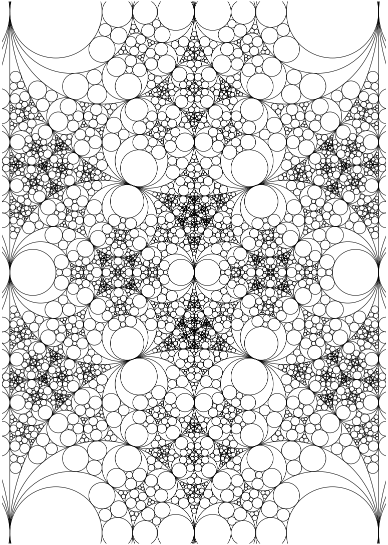


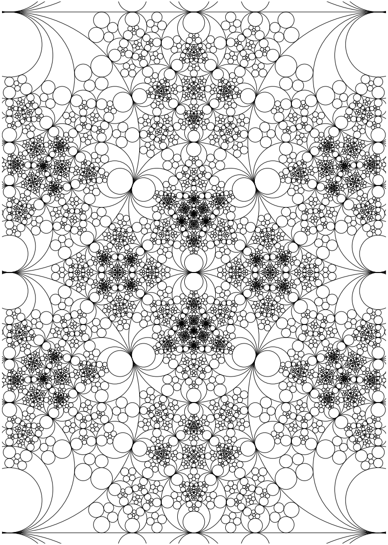
Figure: The density $f_B(z)$ shown from two angles (f_A is f_B rotated by 90°).

Another example: Cubeoctahedral continued fractions over $\mathbb{Q}(\sqrt{-2})$



Super-packings





Inversions in the circles of the previous figure generate a discrete group Γ of isometires of hyperbolic three-space (of finite covolume), reflections in the sides of an ideal, right-angled cubeoctahedron. The cubeoctahedral reflection group Γ is the kernel of the map

$$PGL_2(\mathbb{Z}[\sqrt{-2}]) \rtimes \langle \mathfrak{c} \rangle \to PGL_2(\mathbb{Z}[\sqrt{-2}]/(2)),$$

similar to the Gaussian situation described earlier.

All of the results described earlier have analogues here (dual pair of dynamical systems, invertible extension and invariant measures, etc.). In particular (with 12 sequences of approximations corresponding to the vertices of the cubeoctahedron):

If $(p, q) = 1$ is such that

$$|z_0 - p/q| < \frac{C}{|q|^2}, \; C = \frac{2\sqrt{2}}{1+\sqrt{2}+\sqrt{3}},$$

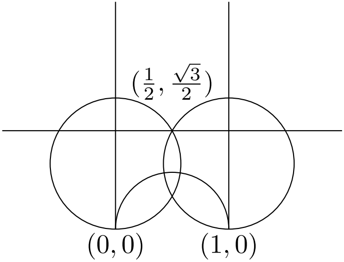
then p/q is a convergent to z_0. Moreover, C is the largest constant possible.

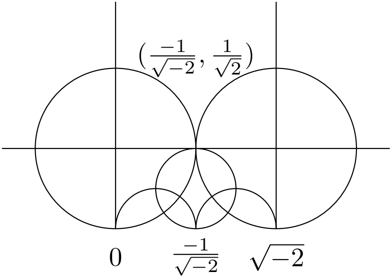
There is a simple geometric argument for the following facts (first proved by L. Ford and O. Perron respectively).

Every $z \in \mathbb{C}$ has infinitely many approximations p/q such that

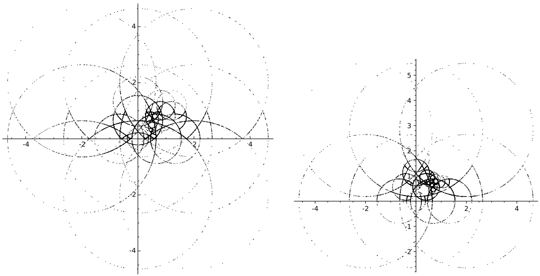
$$|z - p/q| \leq C/|q|^2$$

where $C = 1/\sqrt{3}$ ($\mathbb{Q}(\sqrt{-1})$) or $C = 1/\sqrt{2}$ ($\mathbb{Q}(\sqrt{-2})$). These are the best constants as witnessed by $\frac{1+\sqrt{-3}}{2}$ and $\frac{1+i}{\sqrt{2}}$.

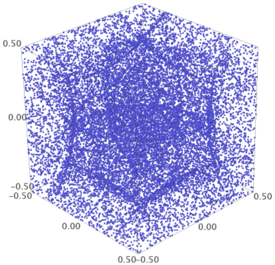




The number $z \in \mathbb{C}$ is badly approximable (over $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$) iff its right-angled orbit is bounded away from the fixed points. Tagged rational binay Hermitian forms have finite orbit under the right-angled algorithms. In particular, zeros of anisotropic forms are badly approximable.



[Me working on the end of my thesis.]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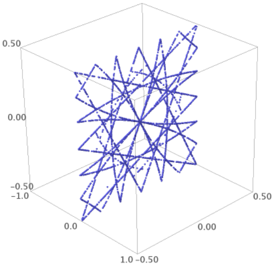
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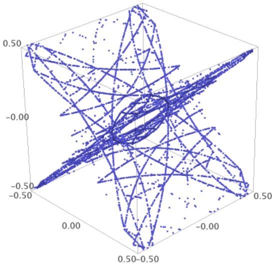
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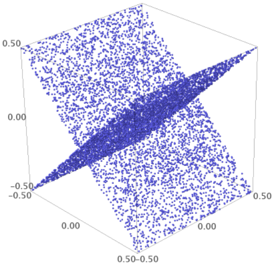
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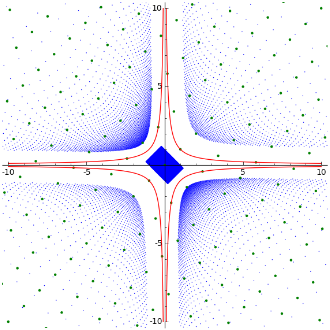
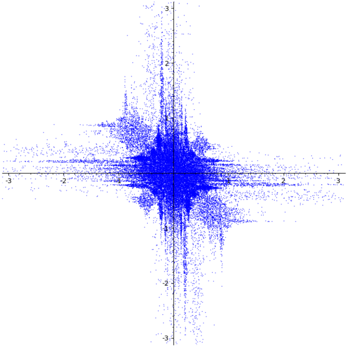
Ch. 7 example: $\mathbb{Q}(\sqrt{2})$ (I didn't prove anything)





$$b_n = \lfloor (n+1)x \rfloor - \lfloor nx \rfloor, \; x \in [0,1] \backslash \mathbb{Q},$$
$$\xi = [a_0; a_1, \ldots], \; (b_n = 0 \mapsto 22, \; b_n = 1 \mapsto 11),$$
ξ transcendental (stammering continued fraction)

E.g.

$$\xi = \sum_{i=0}^{\infty} (-1)^{i+1} \left(3 - \frac{m_{i+2}}{m_i m_{i+1}} \right),$$
$$m_{i+3} = 3 m_{i+2} m_{i+1} - m_i, \; (m_1, m_2, m_3) = (5, 13, 194).$$

Sums over Markoff numbers:

$$\sum_{m \in \mathcal{M}} 3 - \frac{\sqrt{9m^2 - 4}}{m} = \frac{7 - \sqrt{5} - \sqrt{8}}{2},$$

a case of Mcshane's identity

$$\sum_{\gamma} \frac{1}{1 + e^{l(\gamma)}} = \frac{1}{2},$$

(sum over simple closed geodesics on a once-punctured torus).

Finally, the end!