Examples of badly approximable vectors over number fields

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Let $F/\mathbb{Q}$ be a number field of degree $r+2s$, where r and s are the number of real embeddings $F \to \mathbb{R}$ and conjugate pairs of complex embeddings $F \to \mathbb{C}$ respectively.

F is a subfield of $F \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$ and its ring of integers $\mathcal{O}_F$ is a discrete subring.

Let S be the collection of real embeddings along with a choice of complex embedding from each conjugate pair, so that that $a \in F$ is identified with the tuple $(\sigma(a))_{\sigma \in S}$ in the isomorphism above, i.e. if $F = \mathbb{Q}(\alpha)$ and the minimal polynomial of α over $\mathbb{Q}$ is $m(x)$, then

$$F \otimes \mathbb{R} = \frac{\mathbb{Q}[x]}{(m(x))} \otimes \mathbb{R} = \prod_{\substack{\sigma \\ \text{real}}} \frac{\mathbb{R}[x]}{(x-\sigma(\alpha))} \prod_{\substack{\{\sigma,\bar{\sigma}\} \\ \text{complex}}} \frac{\mathbb{R}[x]}{(x^2-(\sigma(\alpha)+\bar{\sigma}(\alpha))x+\sigma(\alpha)\bar{\sigma}(\alpha))}.$$

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$$F \otimes \mathbb{R} = \frac{\mathbb{Q}[x]}{(m(x))} \otimes \mathbb{R} = \prod_{\substack{\sigma \\ \text{real}}} \frac{\mathbb{R}[x]}{(x - \sigma(\alpha))} \prod_{\substack{\{\sigma, \bar{\sigma}\} \\ \text{complex}}} \frac{\mathbb{R}[x]}{(x^2 - (\sigma(\alpha) + \bar{\sigma}(\alpha))x + \sigma(\alpha)\bar{\sigma}(\alpha))}.$$

General question:

How well can $\mathbf{z} \in \mathbb{R}^r \times \mathbb{C}^s$ be approximated by elements of F?

We will measure the approximation of $\mathbf{z} = (z_\sigma)_\sigma$ by $p/q \in F$ with

$$\max_\sigma\{|\sigma(q)|\} \max_\sigma\{|\sigma(q)z_\sigma - \sigma(p)|\}.$$

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We say $\mathbf{z}$ is *badly approximable* if there exists $C' > 0$ such that

$$\max_\sigma\{|\sigma(q)|\} \max_\sigma\{|\sigma(q) z_\sigma - \sigma(p)|\} \geq C'$$

for all $p/q \in F$.

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[If $\mathbf{z}$ is badly approximable as defined above, then

$$\max_\sigma\{|z_\sigma - \sigma(p/q)|\} \geq C' / \max_\sigma\{|\sigma(q)|\}^2,$$

and the converse holds for $(r, s) = (1, 0)$, $(0, 1)$, $(2, 0)$, $(0, 2)$.]

The measure of approximation introduced above has the following properties.

Dirichlet-type

There exists $C > 0$ such that any $\mathbf{z} \notin F$ has infinitely many rational approximations p/q with

$$\max_{\sigma}\{|\sigma(q)|\} \max_{\sigma}\{|\sigma(q)z_\sigma - \sigma(p)|\} \leq C.$$

Roth-type

For any algebraic $\mathbf{z} \notin F$ (each z_σ algebraic) and any $\epsilon > 0$, there exists $C' > 0$ such that

$$\max_{\sigma}\{|\sigma(q)|\}^{1+\epsilon} \max_{\sigma}\{|\sigma(q)z_\sigma - \sigma(p)|\} \geq C'$$

for any $p/q \in F$.

The set of badly approximable vectors has measure zero but full Hausdorff dimension.

*M. Einsiedeler, A. Ghosh, and B. Lyttle: The set of badly approximable vectors is "winning" (even when restricted to curves) in the setting above.
*D. Kleinbock and T. Ly: The set of badly approximable vectors is "$\mathcal{H}$-absolute winning" (even when restricted to curves and some fractals) in the setting above.
*T. Hattori: Proved Dirichlet-type theorems for real quadratic and complex quartic fields - infinitely many solutions to

$$\|\mathbf{z} - p/q\|_1 \leq C/\sqrt{H(q)},$$

and gave examples of badly approximable vectors.
*R. Quême: Proved Dirichlet-type theorems - infinitely many solutions to

$$\|q\|_{1'} \cdot \|q\mathbf{z} - p\|_{1'} \leq C,\ \ N(\mathbf{z} - p/q) \leq C/N(q)$$

where $N(\mathbf{z})$ is the extension of the absolute value of the field norm.
*E. Burger: Diophantine approximation over S-integers and examples of badly approximable linear systems using

$$h_S(\mathbf{x}, \mathbf{y})^N \prod_{v \in S} |A_v\mathbf{x} - \mathbf{y}|_v^M,\ \ \mathbf{x} \in \mathcal{O}_F^N,\ \ ,\mathbf{y} \in \mathcal{O}_F^M,\ \ A_v \in Mat_{M\times N}(F_v).$$

*W. Schmidt: Dirichlet-type theorem (for $\prod_\sigma \sigma(K) \subseteq \mathbb{R}^r \times \mathbb{C}^s$) and the Subspace theorem for number fields (from which the above Roth-type theorem can be deduced).

*S. G. Dani: Characterization of badly approximable systems of linear forms in terms of bounded trajectories in $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.

Let $G = SL_2(F \otimes \mathbb{R}) \cong SL_2(\mathbb{R})^r \times SL_2(\mathbb{C})^s$ and let $K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s$ be a maximal compact subgroup.

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The symmetric space $G/K \cong (\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$ is a product of hyperbolic 2- and 3-spaces (upper half-space model).

[$SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}$ via fractional linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

and the stabilizer of $z = i$ is $SO_2(\mathbb{R})$. Similarly, $SL_2(\mathbb{C})$ acts transitively on $\mathcal{H}^3 = \{\zeta = z + tj \in \mathbb{H} : t > 0\}$ via fractional linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \zeta = (a\zeta + b)(c\zeta + d)^{-1}$$

and the stabilizer of $\zeta = j$ is $SU_2(\mathbb{C})$.]

$\Gamma = SL_2(\mathcal{O}_F)$ is a non-uniform lattice in G - a discrete subgroup such that $\Gamma\backslash G$ is not compact but has finite volume.

We will be interested in the locally symmetric spaces $\Gamma\backslash G/K$.

For example, we have the modular surface $SL_2(\mathbb{Z})\backslash\mathcal{H}^2$, Bianchi orbifolds $SL_2(\mathcal{O}_F)\backslash\mathcal{H}^3$, and Hilbert modular surfaces $SL_2(\mathcal{O}_F)\backslash(\mathcal{H}^2)^2$ for $(r, s) = (1, 0)$, $(0, 1)$, and $(2, 0)$ respectively.

Dani correspondence

The vector $\mathbf{z}$ is badly approximable over F if and only if the trajectory

$$\omega_{\mathbf{z}}(t) = \left\{ \Gamma \cdot \left(\begin{pmatrix} 1 & z_\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{\sigma \in S} \cdot K : t \geq 0 \right\}$$

is bounded in $\Gamma \backslash G / K$

This follows from Mahler's compactness criterion (tailored here to our current needs):

Mahler's criterion

A subset $\Omega \subseteq SL_2(F \otimes \mathbb{R})$ is precompact modulo Γ if and only if the $\mathcal{O}_F$-lattices spanned by the rows of elements of $\omega \in \Omega$ have no arbitrarily short vectors, i.e. there exists $\epsilon > 0$ such that

$$\inf\{\|(p\ q)\omega\| : (0,0) \neq (p,q) \in \mathcal{O}_F^2,\ \omega \in \Omega\} \geq \epsilon.$$

The rest of this talk will be concerned with getting ahold of some obvious bounded trajectories $\omega_{\mathbf{z}}$: those asymptotic to compact subspaces of $\Gamma\backslash G/K$.

The compact subspaces we pursue are those associated to the orthogonal and unitary groups of anisotropic quadratic and Hermitian forms.

Let F be a number field and

$$Q(x,y) = Ax^2+Bxy+Cy^2 = (x\ y)\left(\begin{array}{cc} A & B/2 \\ B/2 & C \end{array}\right)\binom{x}{y},\ A,B,C \in F,$$

an F-rational binary quadratic form.

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Q is *totally indefinite* if $\det(Q) \in F$ is negative under every real embedding.

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Denote by Q_σ the form obtained by applying σ to the coefficients of Q:

$$Q_\sigma = \begin{pmatrix} \sigma(A) & \sigma(B/2) \\ \sigma(B/2) & \sigma(C) \end{pmatrix}.$$

The product of projective spaces $P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$ sits naturally in the boundary of $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$, and we identify $\mathbf{z} = (z_\sigma)_{\sigma \in S} \in \mathbb{R}^r \times \mathbb{C}^s$ with the tuple $([z_\sigma : 1])_\sigma$.

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Let

$$Z(Q_\sigma) = \{[p : q] \in P^1(\mathbb{R}) \text{ or } P^1(\mathbb{C}) : Q_\sigma(p, q) = 0\}$$

be the zero set of Q_σ and $Z(Q) = \prod_\sigma Z(Q_\sigma)$ be their product, a finite set of cardinality 2^{r+s} for totally indefinite Q.

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Note that $g^{-1} \cdot Z(Q) = Z(Q^g)$ for $g = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \right)_\sigma \in G$, where the action on the left is the Möbius/isometric action

$$g \cdot \mathbf{z} = g \cdot (z_\sigma)_\sigma = \left(\frac{a_\sigma z_\sigma + b_\sigma}{c_\sigma z_\sigma + d_\sigma} \right)_\sigma$$

and the action on the right is change of variable

$$Q^g = g^t Q g = (Q_\sigma(a_\sigma x + b_\sigma y, c_\sigma x + d_\sigma y))_\sigma.$$

Let

$$SO(Q, F\otimes\mathbb{R}) = \{g \in G : g^t Q g = Q\},\ SO(Q, \mathcal{O}_F) = \Gamma \cap SO(Q, F\otimes\mathbb{R}),$$

be the orthogonal groups of Q over $F \otimes \mathbb{R}$ and $\mathcal{O}_F$ respectively.

The following is well-known (and can be proved via Mahler's criterion):

If Q is anisotropic, then $SO(Q, \mathcal{O}_F)\backslash SO(Q, F\otimes\mathbb{R}) \subseteq \Gamma\backslash G$ and $\Gamma \cdot SO(Q, F\otimes\mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K$ are compact for any $g \in G$.

In particular, this implies:

If Q is totally indefinite and anisotropic, then $\prod_\sigma L_\sigma \bmod \Gamma$ is a compact totally geodesic subspace of $\Gamma\backslash G/K$, where L_σ is the line in $\mathcal{H}^2$ or $\mathcal{H}^3$ with endpoints $Z(Q_\sigma)$.

Totally indefinite anisotropic binary Hermitian forms (CM fields)

Let F/E be a CM field, (E totally real, F/E imaginary quadratic) and

$$\begin{aligned} H(z,w) =& Az\bar{z} + B\bar{z}w + \overline{B}z\bar{w} + Cw\bar{w} = (\bar{z}\ \bar{w}) \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \\ & A, C \in E, B \in F, \end{aligned}$$

an F-rational binary Hermitian form. [The overline is “complex conjugation”, the non-trivial automorphism of F/E.]

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be the zero set of H_σ and $Z(H) = \prod_\sigma Z(H_\sigma)$ be their product.

[If $A \neq 0$, then
$$Z(H_\sigma) = \{[z : 1] : |z + \sigma(B/A)|^2 = -\det(H_\sigma)/|\sigma(A)|^2\},$$
a circle in $\mathbb{C}$ if H_σ is indefinite. Hence $Z(H) \cong (S^1)^s$ is a product of circles for totally indefinite H.]

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be the unitary groups of H over $F \otimes \mathbb{R}$ and $\mathcal{O}_F$ respectively.

The following is well known (and can be proved with Mahler's criterion):

If H is anisotropic, then $SU(H, \mathcal{O}_F)\backslash SU(H, F \otimes \mathbb{R}) \subseteq \Gamma\backslash G$ and $\Gamma \cdot SU(H, F \otimes \mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K$ are compact for any $g \in G$.

In particular, this implies:

If H is totally indefinite and anisotropic, then $\prod_\sigma P_\sigma$ mod Γ is a compact totally geodesic subspace of $\Gamma\backslash G/K$, where P_σ is the plane in $\mathcal{H}^3$ with boundary $Z(H_\sigma)$.

Let Q be totally indefinite anisotropic F-rational binary quadratic form and $\mathbf{z} \in Z(Q) \subseteq \mathbb{R}^r \times \mathbb{C}^s$. Then $\mathbf{z}$ is badly approximable.

Similarly, if F is a CM field and $\mathbf{z} \in Z(H) \subseteq \mathbb{C}^s$ is a zero of a totally indefinite anisotropic F-rational binary Hermitian form, then $\mathbf{z}$ is badly approximable.

Example 1: $Q = x^2 - (2-\sqrt{2})y^2$ is anisotropic and totally indefinite over $\mathbb{Q}(\sqrt{2})$, so the four vectors

$$\left(\pm\sqrt{2-\sqrt{2}}, \pm\sqrt{2+\sqrt{2}}\right) \in \mathbb{R}^2$$

are badly approximable.

Example 2: $H = |z|^2 - 3|w|^2$ is anisotropic and totally indefinite over $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$, so every vector in the torus

$$\{(\sqrt{3}\cos s + i\sqrt{3}\sin s, \sqrt{3}\cos t + i\sqrt{3}\sin t) : s, t \in [0, 2\pi)\} \subseteq \mathbb{C}^2$$

is badly approximable.

Proof

If $\mathbf{z} \in Z(Q)$ (resp. $Z(H)$), then the trajectory $\omega_{\mathbf{z}}(t)$ is asymptotic to the product of lines $\prod_\sigma L_\sigma$ (resp. the product of planes $\prod_\sigma P_\sigma$).

Modulo Γ, $\prod_\sigma L_\sigma$ (resp. $\prod_\sigma P_\sigma$) is compact.

Therefore $\omega_{\mathbf{z}}(t)$ mod Γ is bounded in $\Gamma\backslash G/K$.

There is also an elementary proof à la Liouville that zeros of totally indefinite anisotropic integral binary quadratic and Hermitian forms are badly approximable. Suppose J is such a form and that $J(\mathbf{z}, 1) = 0$. If $\max_\sigma\{|z_\sigma - \sigma(p/q)|\} \leq 1$, then by the mean value theorem

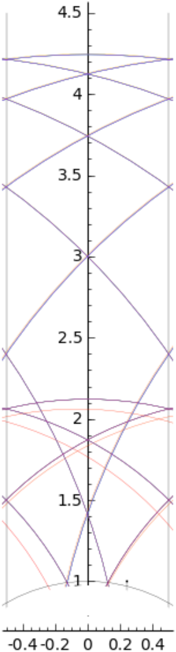
$$|J_\sigma(\sigma(p/q), 1)| = |J_\sigma(z_\sigma, 1) - J_\sigma(\sigma(p/q), 1)| \leq c_\sigma |z_\sigma - \sigma(p/q)|$$

for some $c_\sigma > 0$. Since J is integral and anisotropic,

$$\max_\sigma\{|\sigma(q)^2 J_\sigma(\sigma(p/q), 1)|\} \geq c'$$

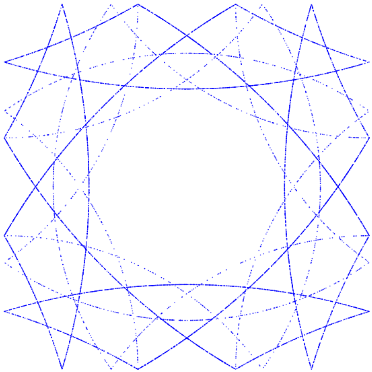
say for $c' = \min\{\max_\sigma\{|\sigma(a)|\} : 0 \neq a \in \mathcal{O}_F\}$. Hence for some σ_0,

$$\max_\sigma\{|\sigma(q)|\} \max_\sigma\{|\sigma(q) z_\sigma - \sigma(p)|\} \geq c_{\sigma_0}/c'.$$



Over $\mathbb{Q}$, we recover the fact that quadratic irrationals are badly approximable. This is usually demonstrated using continued fractions, but here we rely on the correspondence between (equivalence classes of) indefinite (anisotropic, integral) binary quadratic forms and closed geodesics on the modular surface.

Left is the trajectory aimed at $3\sqrt{2} - 4 = [0; \overline{4, 8}]$. The trajectory is bounded, asymptotic to the closed geodesic associated to the form $x^2 + 8xy - 2y^2$.



Orbit of $\mathbf{z} = \sqrt{3}e^{2\pi i/5}$ for a continued fraction algorithm over $F = \mathbb{Q}(\sqrt{-1})$, indicating the contstrained and badly approximable behavior of $\omega_{\mathbf{z}}(t)$.

Over an imaginary quadratic field $F = \mathbb{Q}(\sqrt{-d})$, quadratic irrationals are badly approximable (corresponding to closed geodesics in the Bianchi orbifolds), but we also have zeros of indefinite (anistropic, integral) binary Hermitian forms, associated to compact totally geodesic surfaces. Proofs can be given using (nearest integer) continued fractions when $\mathcal{O}_F$ is Euclidean $(d = 1, 2, 3, 7, 11)$.

We would like to note that the $Z(H)$ contain many algebraic vectors, which we can parameterize as follows.

Choose real algebraic numbers $u_\sigma \in [-2,2]$, a totally positive $t \in E \backslash N_E^F(F)$, and any $s \in F$. Then

$$\mathbf{z} = (z_\sigma)_\sigma,\ z_\sigma = \sigma(s) + \sqrt{\sigma(t)}\,\frac{u_\sigma \pm \sqrt{u_\sigma^2 - 4}}{2},$$

are the algebraic badly approximable vectors associated to Hermitian forms.

Question

Are $Z(Q)$, $Z(H)$ the only badly approximable algebraic vectors?

This is not known even in the case $(r,s) = (1,0)$.

Thank you for your attention,

and thanks to the AMS and everyone involved in the MRC!

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