EXAMPLES OF BADLY APPROXIMABLE VECTORS OVER NUMBER FIELDS

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Abstract. We consider approximation of vectors $\mathbf{z} \in F \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$ by elements of a number field F and construct examples of badly approximable vectors. These examples come from compact subspaces of $SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes \mathbb{R})$ naturally associated to (totally indefinite, anisotropic) F-rational binary quadratic and Hermitian forms, a generalization of the well-known fact that quadratic irrationals are badly approximable over $\mathbb{Q}$.

Introduction

A number field F of degree $r + 2s$ embeds naturally in the product of its Archimedean completions $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$. Given a vector $\mathbf{z} = (z_1, \ldots, z_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^s$, one can ask how well $\mathbf{z}$ can be approximated by elements of F. Following [EGL] and [KL], we will measure the quality of approximation by

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\}, \ p/q \in F, \ p, q \in \mathcal{O}_F,$$

where p_i and q_i are the images of p and q under $r + s$ inequivalent embeddings $\sigma_i : F \to \mathbb{C}$ and $|\cdot|$ is the usual absolute value in $\mathbb{R}$ or $\mathbb{C}$. The measure above is meaningful in the sense that all irrational vectors have infinitely many "good" approximations as demonstrated by the following Dirichlet-type theorem.

Theorem 1 (cf. [Q], Theorem 1). *There is a constant C depending only on F such that for any $\mathbf{z} \notin F$*

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \leq C$$

has infinitely many solutions $p/q \in F$.

In what follows, we will give some explicit examples showing that the above theorem fails if the constant is decreased, i.e. there are *badly approximable* vectors, $\mathbf{z}$ such that there exists $C' > 0$ with

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \geq C'$$

for all $p/q \in F$. Our examples come from "obvious" compact totally geodesic subspaces of

$$SL_2(\mathcal{O}_F)\backslash(\mathbb{H}^2)^r \times (\mathbb{H}^3)^s,$$

where $\mathbb{H}^n$ is hyperbolic n-space. Namely, these examples are associated to totally indefinite anisotropic F-rational binary quadratic forms over any number field (Proposition 1) and totally indefinite anisotropic F-rational binary Hermitian forms over CM fields (Proposition 2). Among the examples are algebraic vectors, i.e. vectors whose entries generate a non-trivial finite extension of F, including non-quadratic vectors in the CM case (Corollary 1). This is interesting in light of the following variation on Roth's theorem (which can be deduced from the Subspace Theorem for number fields).

Theorem 2 (cf. [S1], Theorem 3). *Suppose* $\mathbf{z} \notin F$ *has algebraic coordinates. Then for all* $\epsilon > 0$, *there exists a constant* $C' > 0$ *depending on* $\mathbf{z}$ *and* ϵ *such that*

$$\max_i\{|q_i|\}^{1+\epsilon} \max_i\{|q_i z_i - p_i|\} \geq C'$$

for all $p/q \in F$.

The "linear forms" notion of badly approximable defined above implies

$$\max_i\{|z_i - p_i/q_i|\} \geq \frac{C'}{\max_i\{|q_i|^2\}} \text{ for all } p/q \in F,$$

which is perhaps the first notion of badly approximable that comes to mind. The two notions are equivalent when F has only one infinite place (i.e. $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$) since the absolute value is multiplicative. (The notions are also equivalent for real quadratic and complex quartic F, [KL] Proposition A.2.) However, in larger number fields it seems that some choice must be made and the linear choice appears naturally in the proof of Theorem 3.

Simultaneous approximation in this sense seems natural and has been explored by various authors, e.g. [EGL], [Ha], [KL], [Q], [S1],[B]. Among known facts, we note that the set of badly approximable vectors has Lebesgue measure zero, full Hausdorff dimension, and is even "winning" when restricted to curves and various fractals in $\mathbb{R}^r \times \mathbb{C}^s$ ([EGL], [KL], [EK]).

Finally, we note that there is an elementary proof that our examples are badly approximable, along the lines of Liouville's theorem, which we give at the end of the paper.

Notation and outline

To fix some notation and conventions, we identify $SL_2(\mathbb{C})/SU_2(\mathbb{C})$ with the upper half-space model of three-dimensional hyperbolic space, $\mathbb{H}^3 = \{\zeta = z + tj : z \in \mathbb{C}, 0 < t \in \mathbb{R}\}$ (inside the Hamiltonians), via the action

$$g \cdot \zeta = (a\zeta + b)(c\zeta + d)^{-1},\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}),$$

and $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ with the upper half-plane, $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}$, via the action

$$g \cdot z = \frac{az + b}{cz + d},\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Let F be a number field of degree $r+2s$, where r and s are the number of real and complex places respectively, and let $n = r+s$. We are interested in the locally symmetric space $\Gamma\backslash G/K$ where $\Gamma = SL_2(\mathcal{O}_F)$ acts by left multiplication on $G = SL_2(F \otimes \mathbb{R}) \cong SL_2(\mathbb{R})^r \times SL_2(\mathbb{C})^s$ and $K \cong SO_2(\mathbb{R})^r \times SU_2(\mathbb{C})^s$ is a maximal compact subgroup of G. Let $\{\sigma_i\}_{i=1}^n$ be the set of real embeddings along with a choice of one complex embedding from each conjugate pair, so that Γ acts diagonally via $\{\sigma_i\}_i$ in the isomorphism above.

We will consider products of lines $\prod_i L_i \subseteq (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$ for any F (respectively products of planes $\prod_i P_i \subseteq (\mathbb{H}^3)^n$ in the CM case) whose image modulo Γ is compact, along with geodesic trajectories $\Omega_{\mathbf{z}} \cdot K \subseteq (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$ "aimed" at points $\mathbf{z} = (z_i)_i \in \mathbb{R}^r \times \mathbb{C}^s \subseteq (\partial\mathbb{H}^2)^r \times (\partial\mathbb{H}^3)^s$, where

$$\Omega_{\mathbf{z}} = \left\{ \left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}, \ldots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right) : 0 \leq t \in \mathbb{R} \right\}.$$

If each $z_i \in \partial L_i$ (respectively $z_i \in \partial P_i$) then the trajectory $\Omega_{\mathbf{z}}$ is asymptotic to $\prod_i L_i$ (respectively $\prod_i P_i$) and therefore bounded modulo Γ. The Dani correspondence of the next section tells us such $\mathbf{z}$ are badly approximable.

Dani correspondence

The results of this section are taken from [EGL]. Theorem 3 is a variation of [D], Theorem 2.20, tailored to simultaneous approximation.

Theorem 3 ([EGL], Proposition 3.1)**.** *The vector $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^r \times \mathbb{C}^s$ is badly approximable over F if and only if the geodesic trajectory*

$$\Gamma \cdot \Omega_{\mathbf{z}} \cdot K \subseteq \Gamma \backslash (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$$

is bounded, where

$$\Omega_{\mathbf{z}} = \left\{ \left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}, \dots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right) : 0 \le t \in \mathbb{R} \right\}.$$

This follows in a straight-forward fashion from the following version of Mahler's compactness criterion, stated here for SL_2.

Theorem 4 ([EGL], Theorem 2.2)**.** *A subset $\Gamma \cdot \Omega \subseteq \Gamma \backslash SL_2(F \otimes \mathbb{R})$ is precompact if and only if there exists $\epsilon > 0$ such that*

$$\max\{\max_i\{|z_i|\}, \max_i\{|w_i|\}\} \ge \epsilon, \ \ (\mathbf{z}, \mathbf{w}) = (q, p)\omega,$$

for all $(0,0) \neq (q,p) \in \mathcal{O}_F^2$ and $\omega \in \Omega$. In other words, the two-dimensional $\mathcal{O}_F$-modules in $(F \otimes \mathbb{R})^2$ spanned by the rows of $\omega \in \Omega$ do not contain arbitrarily short vectors.

An obvious way to obtain bounded trajectories in Theorem 3 is to consider those asymptotic to compact totally geodesic subspaces of $\Gamma \backslash (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$, which we do in the next two sections.

Totally indefinite binary quadratic forms

As above, let F be a number field, $F \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, $\Gamma = SL_2(\mathcal{O}_F)$, and let

$$Q(x,y) = (x \ y) \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax^2 + Bxy + Cy^2, \ \ A, B, C \in F$$

be an F-rational binary quadratic form with determinant $\Delta(Q) = AC - B^2/4$. We say Q is *totally indefinite* if $\sigma(\Delta) < 0$ for all real embeddings $\sigma : F \to \mathbb{R}$ and that Q is *anisotropic* if it has no non-trivial zeros in F^2. Note that Q is anisotropic if and only if $-\Delta(Q)$ is not a square in F. Let Q_i be the form obtained by applying σ_i to the coefficients of Q, denote by $Z_i(Q)$ the zero set of Q_i

$$Z_i(Q) = \{[z : w] \in P^1(\mathbb{C}) \text{ or } P^1(\mathbb{R}) : Q_i(z,w) = 0\},$$

and let $Z(Q) = \prod_i Z_i(Q)$ (a finite set of cardinality 2^n for totally indefinite Q).

The group Γ acts on binary quadratic forms by change of variable

$$Q^g(x,y) = (g^t Q g)(x,y) = Q(ax + by, cx + dy)$$

and also on $P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$ diagonally by linear fractional transformations

$$g \cdot ([z_1 : w_1], \dots, [z_n : w_n]) = ([a_1 z_1 + b_1 w_1 : c_1 z_1 + d_1 w_1], \dots, [a_n z_n + b_n w_n : c_n z_n + d_n w_n]),$$

where $a_i = \sigma_i(a)$ and similary for b_i, c_i, and d_i. These actions are compatible in the sense that $g^{-1} \cdot Z(Q) = Z(Q^g)$. Without further remark, we identify $\mathbb{R}^r \times \mathbb{C}^s$ with a subset of $P^1(\mathbb{R})^r \times P^1(\mathbb{C})^s$ via $(z_i)_i \mapsto ([z_i : 1])_i$.

The following is a generalization of the fact that quadratic irrationals are badly approximable over $\mathbb{Q}$ ($r = 1$, $s = 0$), which is usually demonstrated via continued fractions (e.g. [K], Theorem 28). We should note that these examples can also be deduced from Theorem 6.4 of [B] (with S the set of infinite places and $N = 1$).

Proposition 1. *Let Q be a totally indefinite anisotropic F-rational binary quadratic form over a number field F. Then any vector $\mathbf{z} \in Z(Q)$ is badly approximable over F.*

First we establish compactness of a subspace associated to Q. There are many references with discussions of compactness for anisotropic arithmetic quotients, e.g. [PR], [R], and [W].

Lemma 1. *Let Q be a totally indefinite F-rational anisotropic binary quadratic form, and let L_i be the line in $\mathbb{H}^2$ or $\mathbb{H}^3$ with endpoints $Z(Q_i)$. Then $\pi(\prod_i L_i)$ is compact in $\Gamma\backslash(\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$, where $\pi : (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s \to \Gamma\backslash(\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$ is the quotient map.*

Proof of Lemma 1. Without loss of generality, suppose Q has integral coefficients, $A, B, C \in \mathcal{O}_F$. Compactness of

$$SO(Q, \mathcal{O}_F)\backslash SO(Q, F \otimes \mathbb{R}) \subseteq \Gamma\backslash SL_2(F \otimes \mathbb{R}),$$

is a consequence of Mahler's compactness criterion as follows. For $g \in SO(Q, F \otimes \mathbb{R})$ and any $(0, 0) \neq (\alpha, \beta) \in \mathcal{O}_F^2$, the quantity $\max_i\{|\sigma_i(Q^g(\alpha, \beta))|\}$ is bounded away from zero because

$$0 \neq Q^g(\alpha, \beta) = Q(\alpha, \beta) \in \mathcal{O}_F,$$

and $\mathcal{O}_F$ is discrete in $F \otimes \mathbb{R}$. By Mahler's criterion and continuity of Q viewed as a function $(\mathbb{R}^r \times \mathbb{C}^s)^2 \to \mathbb{R}^r \times \mathbb{C}^s$, $SO(Q, \mathcal{O}_F)\backslash SO(Q, F \otimes \mathbb{R})$ is precompact.[1] The inclusion above is a closed embedding, hence its image is compact.

To get the result in the locally symmetric space, note that

$$\pi(\prod_i L_i) = \Gamma \cdot SO(Q, F \otimes \mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K,$$

where $g \in SL_2(F \otimes \mathbb{R})$ is any element such that $gK \in \prod_i L_i$, and that $\Gamma\backslash G \to \Gamma\backslash G/K$ is proper. $\square$

Proof of Proposition 1. Let L_i be the line in $\mathbb{H}^2$ or $\mathbb{H}^3$ with ideal endpoints the zeros of Q_i as in the lemma. The stabilizer $Stab_\Gamma(\prod_i L_i)$ acts cocompactly on $\prod_i L_i$ by the lemma above. For $\mathbf{z} \in Z(Q)$ the distance between the geodesic trajectory $\Omega_{\mathbf{z}} \cdot K$ and $\prod_i L_i$ is bounded as their projections are asympototic in each copy of $\mathbb{H}^2$ or $\mathbb{H}^3$. In the quotient $\Gamma\backslash(\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$, the image of $\prod_i L_i$ is compact and therefore $\Gamma \cdot \Omega_{\mathbf{z}} \cdot K$ is bounded in $\Gamma\backslash(\mathbb{H}^2)^r \times (\mathbb{H}^3)^s$. $\square$

Totally indefinite binary Hermitian forms over CM fields

Let F be a CM field (an imaginary quadratic extension of a totally real field E) of degree $2n$ with ring of integers $\mathcal{O}_F$ and let H be an F-rational binary Hermitian form

$$H(z, w) = (\bar{z}\ \bar{w}) \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = Az\bar{z} + \overline{B}z\bar{w} + B\bar{z}w + Cw\bar{w},\ A, C \in E,\ B \in F,$$

[1] Due to choices of left/right actions, this technically shows that $SO(F \otimes \mathbb{R})/SO(\mathcal{O}_F)$ has compact closure in $SL_2(F \otimes \mathbb{R})/\Gamma$. However the left and right coset spaces are homeomorphic.

where the overline is "complex conjugation" (the non-trivial automorphism of F/E). Let H_i be the form obtained by applying σ_i to the coefficients of H (noting that σ_i commutes with complex conjugation). We say H is *totally indefinite* if $\sigma_i(\Delta) < 0$ for all i, where $\Delta = \det(H) = AC - B\overline{B}$. We say H is *anisotropic* if $H(p,q) \neq 0$ for $(p,q) \in F^2 \setminus \{(0,0)\}$. Note that H is anisotropic if and only if $-\Delta$ is not a relative norm, $-\Delta \notin N_E^F(F)$. Denote by $Z_i(H)$ the zero set of H_i,

$$Z_i(H) = \{([z:w] \in P^1(\mathbb{C}) : H_i(z,w) = 0\},$$

a circle in $P^1(\mathbb{C})$, and let $Z(H) = \prod_i Z_i(H)$. When H is totally indefinite, $Z(H) \cong (S^1)^s$ is an s-dimensional torus.

The group $\Gamma = SL_2(\mathcal{O}_F)$ acts on H by change of variable

$$H^g(z,w) = (\bar{g}^t H g)(z,w) = H(az+bw, cz+dw), \; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F),$$

and also on $P^1(\mathbb{C})^n$ diagonally by linear fractional transformations

$$g \cdot ([z_1 : w_1], \ldots, [z_n : w_n]) = ([a_1 z_1 + b_1 w_1 : c_1 z_1 + d_1 w_1], \ldots, [a_n z_n + b_n w_n : c_n z_n + d_n w_n]).$$

where $a_i = \sigma_i(a)$ and similarly for b_i, c_i, and d_i. These actions are compatible in the sense that $g^{-1} \cdot Z(H) = Z(H^g)$. As before, we include $\mathbb{C}^s \hookrightarrow P^1(\mathbb{C})^s$ via $(z_i)_i \mapsto ([z_i : 1])_i$.

The following is a generalization of the fact that zeros of anisotropic binary Hermitian forms are badly approximable over imaginary quadratic fields ($r = 0$, $s = 1$). As in the case of quadratic irrationals over $\mathbb{Q}$, this can be demonstrated with continued fractions when the imaginary quadratic field is Euclidean, $F = \mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. Details for the imaginary quadratic case can be found in [Hi].

Proposition 2. *If $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a zero of the totally indefinite anisotropic F-rational binary Hermitian form H, i.e. $\mathbf{z} \in Z(H)$, then $\mathbf{z}$ is badly approximable.*

As before, we first establish compactness of a subspace associated to H (once again, cf. [PR], [R], or [W]).

Lemma 2. *If H is a totally indefinite anisotropic F-rational binary Hermitian form, then $\pi\left(\prod_i P_i\right)$ is compact in $\Gamma\backslash(\mathbb{H}^3)^n$ where P_i is the geodesic plane in the ith copy of $\mathbb{H}^3$ whose boundary at infinity is the zero set $Z_i(H)$ and $\pi : (\mathbb{H}^3)^n \to \Gamma\backslash(\mathbb{H}^3)^n$ is the quotient map.*

Proof of Lemma 2. Without loss of generality, suppose H has integral coefficients, $A, C \in \mathcal{O}_F$, $B \in \mathcal{O}_E$. Compactness of

$$SU(H, \mathcal{O}_F)\backslash SU(H, F \otimes \mathbb{R}) \subseteq \Gamma\backslash SL_2(F \otimes \mathbb{R}),$$

follows from Mahler's compactness criterion as follows. For $g \in SU(H, F \otimes \mathbb{R})$ and any $(0,0) \neq (\alpha, \beta) \in \mathcal{O}_F^2$, the quantity $\max_i\{|\sigma_i(H^g(\alpha,\beta))|\}$ is bounded away from zero because

$$0 \neq H^g(\alpha, \beta) = H(\alpha, \beta) \in \mathcal{O}_E,$$

and $\mathcal{O}_E$ is discrete in $F \otimes \mathbb{R}$. By Mahler's criterion and continuity of H viewed as a function $(\mathbb{C}^s)^2 \to \mathbb{C}^s$, $SU(H, \mathcal{O}_F)\backslash SU(H, F \otimes \mathbb{R})$ is precompact.[2] The inclusion above is a closed embedding, hence its image is compact.

[2]Once again, due to choices of left/right actions, this technically shows that $SU(H, F \otimes \mathbb{R})/SU(H, \mathcal{O}_F)$ has compact closure in $SL_2(F \otimes \mathbb{R})/\Gamma$. However the left and right coset spaces are homeomorphic.

To get the result in the locally symmetric space, note that $\pi(\prod_i P_i) = \Gamma \cdot SU(H, F \otimes \mathbb{R})g \cdot K \subseteq \Gamma\backslash G/K$ where $g \in SL_2(F \otimes \mathbb{R})$ is any element such that $gK \in \prod_i P_i$, and that $\Gamma\backslash G \to \Gamma\backslash G/K$ is proper. $\square$

Proof of Proposition 2. The distance between the geodesic trajectory $\Omega_{\mathbf{z}} \cdot SU_2(\mathbb{C})^n$ and $\prod_i P_i$ is bounded as their projections are asympototic in each of the n copies of $\mathbb{H}^3$. In the quotient $\Gamma\backslash(\mathbb{H}^3)^n$, the image of $\prod_i P_i$ is compact and therefore $\Gamma \cdot \Omega_{\mathbf{z}} \cdot SU_2(\mathbb{C})^n$ is bounded in $\Gamma\backslash(\mathbb{H}^3)^n$. $\square$

It should be emphasized that the badly approximable product of circles $Z(H) \subseteq \mathbb{C}^n$ contains non-quadratic algebraic vectors, parameterized as follows (cf. [Hi]).

Corollary 1. *Choose real algebraic numbers $\alpha_i \in [-2, 2]$, $1 \le i \le n$, $f \in F$, and a totally positive $e \in E \setminus N_E^F(F)$. Then the vectors*

$$\mathbf{z} = (z_1, \dots, z_n), \;\; z_i = f_i + \sqrt{e_i} \cdot \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4}}{2}$$

are algebraic and badly approximable, where $f_i = \sigma_i(f)$, $e_i = \sigma_i(e)$.

Elementary proofs

Finally, we note that Proposition 1 and 2 have elementary proofs along the lines of Liouville's theorem. Let J be a totally indefinite, anisotropic, integral binary quadratic or Hermitian form. For $\mathbf{z} \in Z(J)$, we have

$$|J_i(p_i/q_i, 1)| = |J(z_i, 1) - J(p_i/q_i, 1)| \le \kappa_i |z_i - p_i/q_i|$$

for some constant $\kappa_i > 0$ depending on z_i and J_i by the mean value theorem. Multiplying by $|q_i|^2$ we have

$$|J_i(p_i, q_i)| \le \kappa_i |q_i(q_i z_i - p_i)|.$$

Because J is anisotropic and integral, for any $0 < \lambda \le \min\{\max_i\{|a_i|\} : 0 \ne a \in \mathcal{O}_F\}$ we have

$$\max_i\{|J_i(p_i, q_i)|\} \ge \lambda.$$

Hence for some i_0 we have

$$\lambda \le \kappa_{i_0} |q_{i_0}(q_{i_0} z_{i_0} - p_{i_0})| \le \kappa_{i_0} \max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\},$$

and $\mathbf{z}$ is badly approximable

$$\max_i\{|q_i|\} \max_i\{|q_i z_i - p_i|\} \ge \lambda \kappa_{i_0}^{-1}.$$

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