A short derivation of the Gauss measure for simple continued fractions

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Let $G = \{(x_1, x_2) \in P^1(\mathbb{R}) \times P^1(\mathbb{R}) \backslash \text{diag.}\}$ interpreted as oriented geodesics of the hyperbolic plane. If $m : P^1(\mathbb{R}) \to P^1(\mathbb{R})$ is defined by $m(x) = \frac{ax+b}{cx+d}$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $\bar{m} : G \to G$, $\bar{m}(x, y) = (m(x), m(y))$, preserves the measure $d\eta(x, y) = \frac{dxdy}{(x-y)^2}$ on G:

$$\begin{aligned}\frac{dm(x)dm(y)}{(m(x) - m(y))^2} &= \frac{dxdy}{\left(\frac{ax+b}{cx+d} - \frac{ay+b}{cy+d}\right)^2} \frac{(ad - bc)^2}{(cx + d)^2(cy + d)^2} \\ &= \frac{(cx + d)^2(cy + d)^2}{(ad - bc)^2(x - y)^2} \frac{(ad - bc)^2}{(cx + d)^2(cy + d)^2} dxdy = \frac{dxdy}{(x - y)^2}\end{aligned}$$

To discuss continued fractions, let $a(x) = 1/x, b(x) = x - 1$ with associated matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and define $t : (0, \infty) \to (0, \infty)$ by

$$t(x) = \begin{cases} a(x) & x \in (0, 1) \\ b(x) & x \in (1, \infty) \end{cases}.$$

Consider the set of geodesics

$$X = \{(x, y) \in (-\infty, -1) \times (0, 1) \cup (-\infty, 0) \times (1, \infty))\}.$$

and define $T : X \to X$ by

$$T(x, y) = \begin{cases} (b(x), b(y)) = (x - 1, y - 1) & y \in (1, \infty) \\ (a(x), a(y)) = (1/x, 1/y) & y \in (0, 1) \end{cases},$$

(see figure below). Then $\pi \circ T = t \circ \pi$ and T is bijective a.e. (T is an invertible extension of t). Hence T preserves the measure η as detailed above. The first return T_0 of T to $X_0 = (-\infty, -1) \times (0, 1)$ is also invertible, given by

$$T_0(x, y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{y} \right\rfloor, \left\{ \frac{1}{y} \right\}\right)$$

and is an invertible extension of the Gauss map (first return of t to $(0, 1)$), $t_0(x) = \{1/x\}$.

The push-forward of η (by projection onto the second coordinate) is

$$d\mu(y) = \begin{cases} dy \int_{-\infty}^{-1} \frac{dx}{(x-y)^2} = \frac{dy}{1+y} & y \in (0,1) \\ dy \int_{-\infty}^{0} \frac{dx}{(x-y)^2} = \frac{dy}{y} & y \in (1,\infty) \end{cases}$$

recovering the (t_0-invariant) Gauss measure μ on $(0,1)$.