# Parameterizing Polygons with Grassmannians

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October 27, 2018

**Abstract**

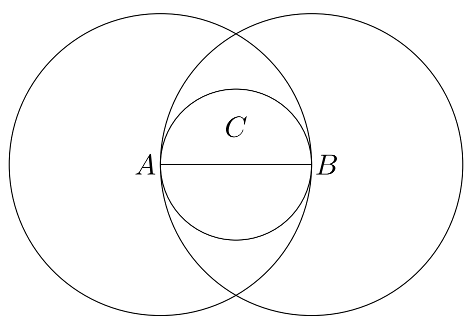
We motivate the construction of a nice moduli space for triangles with a "pillow problem" of Lewis Carroll: Choose three points at random in the plane. What is the probability that the triangle they form is obtuse? [This is a version of a talk given by Jason Cantarella at JMM 2017, essentially contained in [2].]

## 1 Random Triangles

Here is a problem from Lewis Carroll's *Pillow Problems*:

> Choose three points at random in the plane. What is the probability that the triangle they form is obtuse?

Here is a "solution." Let $AB$ be the longest side of the triangle. Then the third vertex $C$ lies in the intersection of the two circles centered at $A$, $B$, and is the triangle is obtuse if and only if $C$ is inside the circle centered at the midpoint of $AB$ (see figure).




The ratio of these areas is $\frac{\pi/2}{2\int_0^1 \sqrt{4-(1+x)^2}dx} = \frac{\pi/2}{4\pi/3-\sqrt{3}} = 0.639382\ldots$. However, repeating the experiment with $AB$ the second longest side, we get $\frac{\pi}{\sqrt{3}+2\pi/3} = 0.821021\ldots$. Taking the coordinates of the vertices $x_i$, $y_i$, $i = 1, 2, 3$, to be IID standard normal, we get an answer of 3/4; see [4].

The problem is ill-posed as the "randomness" of the points in the plane is not specified. What we need is a good moduli space of triangles.

## 2 Parameterizing $n$-gons

Let $v_1, \ldots, v_n \in \mathbb{R}^2$, $e_k = v_k - v_{k-1}$ (indices modulo $n$). It follows that $\sum_k e_k = 0$. Introduce complex coordinates by choosing $w_k^2 = e_k$ (there is a choice of $2^n$ signs). Let $w_k = a_k + ib_k$.

**Proposition 1.** *[3] The complex vector $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ is the "square root" of a polygon if and only if the vectors*

$$a = (a_1, \dots, a_n),\ b = (b_1, \dots, b_n)$$

*are orthogonal of the same length, i.e. $\sum_k a_k b_k = 0$, $\sum_k a_k^2 = \sum_k b_k^2$. If $w$ is the square root of a polygon, then the perimeter of the polygon, $\sum_k |e_k|$, is $\|a\|^2 + \|b\|^2$.*

*Proof.* We have $w_k^2 = a_k^2 - b_k^2 + 2a_k b_k i$. Hence

$$0 = \sum_k e_k = \sum_k w_k^2 = \sum_k (a_k^2 - b_k^2) + i \sum_k 2a_k b_k \Longleftrightarrow \sum_k a_k b_k = 0,\ \sum_k a_k^2 = \sum_k b_k^2.$$

For the second part, we have

$$\sum_k |e_k| = \sum_k |w_k|^2 = \sum_k (a_k^2 + b_k^2) = \|a\|^2 + \|b\|^2.$$

□

Hence the collection of $n$-gons of a fixed perimeter (say $2 = 1^2 + 1^2$), up to translation, is $2^n$-covered by the Stiefel manifold $V_2(\mathbb{R}^n)$ of orthonormal 2-frames in $\mathbb{R}^n$. The following proposition allows us to further quotient by rotations.

**Proposition 2.** *[3] The polygon(s) $P' = (a', b')$ obtained by rotating $P = (a, b)$ by an angle $\theta$ is given by (with choice of signs for each $k$)*

$$a' = a\cos(\theta/2) - b\sin(\theta/2),\ b' = a\sin(\theta/2) + b\cos(\theta/2).$$

*In other words, rotating $P$ by $\theta$ in the plane corresponds to rotating the frame $a$, $b$ by $\theta/2$ within the plane it spans.*

*Proof.* We have

$$e_k' = e^{i\theta} e_k = e^{i\theta} w_k^2 = (\pm e^{i\theta/2} w_k)^2,$$

$$w_k' = \pm e^{i\theta/2} w_k = \pm(a_k + b_k i)e^{i\theta/2} = \pm[(a_k \cos(\theta/2) - b_k \sin(\theta/2)) + (a_k \sin(\theta/2) + b_k \cos(\theta/2))i].$$

□

Hence $n$-gons of a fixed perimeter, up to (orientation preserving) Euclidean isometry, are $2^n$-covered by the Grassmannian of 2-planes in $\mathbb{R}^n$, $G_2(\mathbb{R}^n)$.

**Theorem 1.** *Let*

$$Poly(n, 2) = \{(v_1, \dots, v_n) \in \mathbb{C}^n : \sum_k e_k = 0,\ \sum_k |e_k| = 2\}$$

*be the planar polygons with $n$ labeled vertices and perimeter two.*

*We have an isomorphism*

$$Poly(n, 2)/\mathbb{R}^2 \rtimes SO_2(\mathbb{R}) \cong G_2(\mathbb{R}^n)/\{\pm 1\}^n$$

*mapping the class of $(v_1, \dots, v_n)$ to the class of $(a, b)$. Here $\mathbb{R}^2 \rtimes SO_2(\mathbb{R})$ is the orientation preserving isometry group of the plane, acting diagonally on the vertices $v_k$ and $\{\pm 1\}^n$ acts diagonally and coordinate-wise on $\mathbb{R}^n$ (i.e. the choice of signs in the square root $w_k$).*

## 3 Applications and Extensions

The above discussion gives a "good" moduli space of $n$-gons up to dilation, rotation, and translation. It is compact and comes with a natural metric and probability measure as a quotient of $SO(n)$. So we can talk about things like the probability of a random polygon having such and such a property, or the shortest distance between two polygons.

This can also be extended to space curves using quaternions and complex Grassmannians, and higher dimensional spaces using Clifford algebras[?]. [I have no idea if this is true.]

## 4 Back to the Random Triangle

Our moduli space is $G_2(\mathbb{R}^3) \cong P^2(\mathbb{R})$ (planes in $\mathbb{R}^3$ are determined by their normal lines), and the $SO(3)$-invariant measure on $P^2(\mathbb{R})$ comes from the usual surface area on the sphere. To make this explicit, we introduce the following coordinates for triangles of perimeter two:

$$\begin{aligned} a_1^2 + b_1^2 &= |w_1|^2 = |e_1| = 1 - x^2, \\ a_2^2 + b_2^2 &= |w_2|^2 = |e_2| = 1 - y^2, \\ a_3^2 + b_3^2 &= |w_3|^2 = |e_3| = 1 - z^2. \end{aligned}$$

Given $v = (x, y, z) \in S^2$, choosing orthonormal $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ to form a right-handed orthonormal basis for $\mathbb{R}^3$ tells us that $x$, $y$ and $z$ must satisfy the above, giving us the sphere as a two-fold cover of $G_2(\mathbb{R}^3)$ (and 16-fold cover of our moduli space of triangles).

The right triangles are those satisfying the Pythagorean identity, one of the $xyz$-cyclic permutations of

$$(1 - x^2)^2 + (1 - y^2)^2 = (1 - z^2)^2,$$

i.e. (replacing $z^2 = 1 - x^2 - y^2$)

$$x^2 + x^2y^2 + y^2 = 1, \; x^2 = \frac{1 - y^2}{1 + y^2}.$$

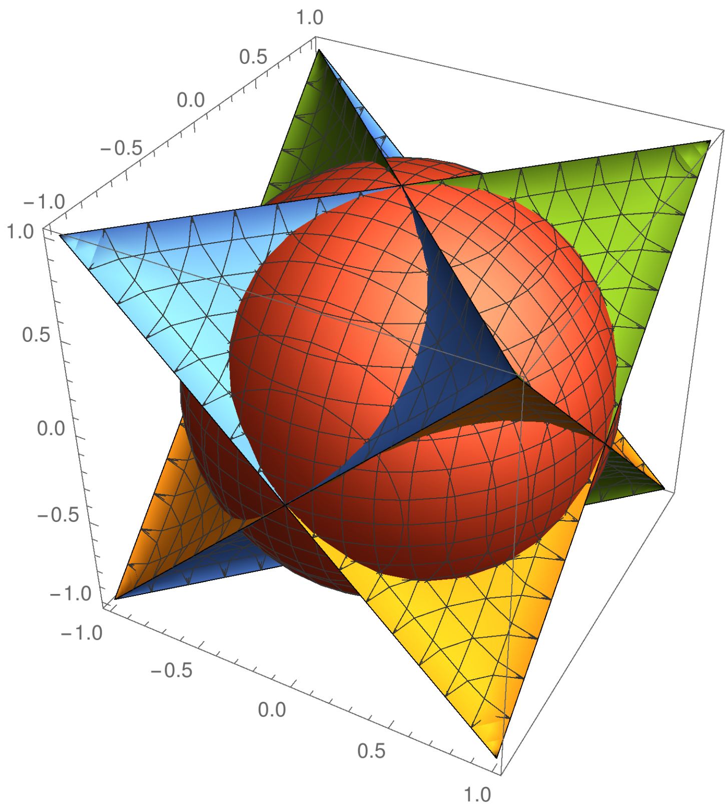
So the locus of right triangles is parameterized by

$$\left( \pm\sqrt{\frac{1 - y^2}{1 + y^2}}, \pm y, \pm y\sqrt{\frac{1 - y^2}{1 + y^2}} \right)$$

and the same with $y$ replaced by $x$ and $z$. Let $R$ be the region on the unite sphere with $x, y, z > 0$ bounded by $z = 0$ and $(1 - x^2)^2 + (1 - y^2)^2 = (1 - z^2)^2$. This region is 1/24 of the obtuse triangles on the sphere(8 choices of $\pm$ 3 right angle equations). The area of the obtuse triangles on the sphere (in cylindrical coordinates, $\rho^2 \sin\phi \, d\phi \, d\theta = dz \, d\theta$ when $\rho = 1$) is therefore

$$\begin{aligned} 24 \iint_R dz d\theta &\overset{Stokes}{=} 24 \int_{\partial R} z d\theta = 0 + 24 \int_0^1 y \sqrt{\frac{1 - y^2}{1 + y^2}} d(\arctan(y/x)) \\ &= \int_0^1 y \sqrt{\frac{1 - y^2}{1 + y^2}} d \left( \arctan \left( y \sqrt{\frac{1 + y^2}{1 - y^2}} \right) \right) = 24 \int_0^1 \left( \frac{2y}{1 + y^4} - \frac{y}{1 + y^2} \right) dy \\ &= 24 \int_0^1 \frac{du}{1 + u^2} - 12 \int_1^2 \frac{du}{u} = 6\pi - 12 \ln 2, \end{aligned}$$

giving a probability (dividing by $4\pi$) of $\frac{3}{2} - \frac{3 \ln 2}{\pi} = 0.838093199-$.




## References

[1] Cantarella, Deguchi, Shonkwiler, *Probability Theory of Random Polygons from the Quaternionic Viewpoint*, Comm. Pure Appl. Math. 67, (2014), no. 10, 1658-1699.

[2] Cantarella, Needham, Shonkwiler, Stewart, *Random triangles and polygons in the plane*, `https://arxiv.org/abs/1702.01027`.

[3] Haussman, Knutson, *Polygon Spaces and Grassmannians*, `https://arxiv.org/pdf/dg-ga/9602012v1.pdf`

[4] Stephen Portnoy, *A Lewis Carroll Pillow Problem: Probability of an Obtuse Triangle*, Statistical Science, Vol. 9, No. 2, 1994, pp. 279-284.

[5] Shonkwiler, *Grassmannians and Random Polygons* (slides), `http://www.math.colostate.edu/~clayton/teaching/DiffGeomMinicourse/polygons.pdf`