James' space: a (counter-)example in Banach spaces

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1 Preliminaries

In this section, we introduce some of the ideas needed to prove desired results about James' space. All of our vector spaces are real. We being with some definitions.

A **(Schauder) basis** in a separable Banach space $(X, \|\cdot\|)$ is a sequence of vectors $e_k \in X$ such that every $x \in X$ can be uniquely written as a convergent series $\sum_{k=1}^{\infty} \alpha_k e_k$ with $\alpha_k \in \mathbb{R}$. We have **canonical projections** $P_n : X \to X$, $P_n(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{n} \alpha_k e_k$ and **coordinate functionals**, $e_n^*(\sum_{k=1}^{\infty} \alpha_k e_k) = \alpha_n$.

[A few remarks. First, bases are ordered and the order matters; there is a notion of an unconditional basis, but James' Space does not have an unconditional basis, so we will not discuss it at any length. Second, not every separable Banach space has a basis (Enflo, 1973), although most familiar examples do. Every Banach space contains a basic sequence (a basis for its closed linear span), but not every Banach space contains an *unconditional* basic sequence (Gowers and Maurey, 1993).]

We now show that the canonical projections and coordinate functionals are continuous, in fact uniformly bounded. We call $K = \sup_n \|P_n\|$ the **basis constant** for the basis e_k and say the basis is **monotone** if $K = 1$.

Proposition 1. *If $(X, \|\cdot\|)$ is a Banach space with basis e_k, the the canonical projections are uniformly bounded*

The proposition follows from the following lemma (noting that $\|P_n(\sum_k \alpha_k e_k)\| \leq \|\{\alpha_k\}_k\|_Y$ using the notation that follows).

Lemma 1. *If $Y = \left\{\{\alpha_k\}_k \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} \alpha_k e_k \text{ converges in } (X, \|\cdot\|)\right\}$ and $\|\{\alpha_k\}_k\|_Y = \sup_n\{\|\sum_{k=1}^{n} \alpha_k e_k\|\}$, then $(Y, \|\cdot\|_Y)$ is a Banach space isomorphic to X.*

Proof. It is clear that $\|\cdot\|_Y$ is a norm. To see that Y is complete, let $\left\{\alpha_k^{(i)}\right\}_k$ be a Cauchy sequence in Y. Then for a fixed l, $\alpha_l^{(i)}$ is Cauchy because

$$\left|\alpha_l^{(i)} - \alpha_l^{(j)}\right| \|e_l\| = \left\|\sum_{k=1}^{l} \left(\alpha_k^{(i)} - \alpha_k^{(i)}\right) e_k - \sum_{k=1}^{l-1} \left(\alpha_k^{(i)} - \alpha_k^{(i)}\right) e_k\right\| \leq 2 \sup_n \left\{\left\|\sum_{k=1}^{n} \alpha_k e_k\right\|\right\} \to 0$$

as $i, j \to \infty$. Hence $\alpha_k^{(n)}$ has a limit, $\lim_n \alpha_k^{(n)} =: \alpha_k$. Let $\epsilon > 0$, j_ϵ such that for $i, j \geq j_\epsilon$ and for every $n > 0$ we have

$$\left\|\sum_{k=1}^{n} \left(\alpha_k^{(i)} - \alpha_k^{(j)}\right) e_k\right\| < \epsilon,$$

and n_ϵ such that for $n, m \geq n_\epsilon$ we have

$$\left\| \sum_{k=n}^{m} \alpha_k^{(j_\epsilon)} e_k \right\| < \epsilon.$$

Then for $j = j_\epsilon$ and $i \to \infty$ we have (for all n)

$$\left\| \sum_{k=1}^{n} \left(\alpha_k - \alpha_k^{(j_\epsilon)} \right) e_k \right\| \leq \epsilon,$$

and for $n, m \geq n_\epsilon$

$$\begin{aligned} \left\| \sum_{k=n}^{m} \alpha_k e_k \right\| &= \left\| \sum_{k=n}^{m} \alpha_k e_k - \sum_{k=n}^{m} \alpha_k^{(j_\epsilon)} e_k + \sum_{k=n}^{m} \alpha_k^{(j_\epsilon)} e_k \right\| \\ &\leq \left\| \sum_{k=1}^{m} \alpha_k e_k - \sum_{k=1}^{m} \alpha_k^{(j_\epsilon)} e_k \right\| + \left\| \sum_{k=1}^{n-1} \alpha_k e_k - \sum_{k=1}^{n-1} \alpha_k^{(j_\epsilon)} e_k \right\| + \left\| \sum_{k=n}^{m} \alpha_k^{(j_\epsilon)} e_k \right\| \\ &\leq 3\epsilon. \end{aligned}$$

Hence $\sum_k \alpha_k e_k$ converges and Y is complete.

The map $S : Y \to X$ given by $S(\{\alpha_k\}_k) = \sum_{k=1}^{\infty} \alpha_k e_k$ is clearly a linear bijection, and is also continuous since

$$\|S(\{\alpha_k\}_k)\| \leq \sup_n \left\{ \left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \right\} = \|\{\alpha_k\}_k\|_Y$$

(note that $\lim_{n\to\infty} \| \sum_{k=1}^{n} \alpha_k e_k\| = \|S(\{\alpha_k\}_k)\|$). By the open mapping theorem, S^{-1} is continuous and there is a K such that $\|\{\alpha_k\}_k\|_Y \leq K \| \sum_{k=1}^{\infty} \alpha_k e_k\|$. □

More generally we have the following (proof omitted).

Proposition 2. *A sequence e_k of a Banach space $(X, \|\cdot\|)$ is a basis if and only if*

1. *$e_k \neq 0$ for each k,*
2. *$X = \overline{\langle e_k \rangle_{\mathbb{R}}}$, and*
3. *there is an M such that for all $n \leq m$ and $\alpha_k \in \mathbb{R}$ we have*

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \leq M \left\| \sum_{k=1}^{m} \alpha_k e_k \right\|.$$

Next we have a characterization of reflexivity for a Banach space with basis.

Theorem 1. *A Banach space $(X, \|\cdot\|)$ with basis e_n is reflexive if and only if the basis is **shrinking**: for every $f \in X^*$ we have*

$$\lim_{n\to\infty} \sup \left\{ |f(x)| : x = \sum_{k=n}^{\infty} \alpha_k e_k, \|x\| = 1 \right\} = 0$$

and

boundedly complete: *if $\alpha_k \in \mathbb{R}$ is such that*

$$\sup_n \left\{ \left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \right\} < \infty,$$

then there is an $x \in X$ such that $x = \sum_{k=1}^{\infty} \alpha_k e_k$.

Proof. Recall that X is reflexive if and only if the unit ball is weakly compact (a consequence of Alaoglu's theorem).

If X is reflexive and has a non-shrinking basis e_k, then there is an $f \in X^*$, $\epsilon > 0$, and sequence $x_n = \sum_{k=p_n}^{\infty} \alpha_k^{(n)} e_k$ with $\|x_n\| = 1$ and $p_n \to \infty$ such that $|f(x_n)| > \epsilon$ for every n. Because X is reflexive, $\{x_n\}$ has a weakly convergent subsequence, $x_{n_i} \overset{\text{wk}}{\to} x$. Along this subsequence, we have $e_k^*(x_{n_i}) \to 0$ and $e_k^*(x_{n_i}) \to e_k^*(x)$ so that $x \equiv 0$ (all of its coordinates are zero), but $\epsilon < |f(x_{n_i})| \to |f(0)| = 0$, a contradiction. Hence a basis for a reflexive space is shrinking.

If X is reflexive and α_n are such that $\sup_n\{\| \sum_{k=1}^{n} \alpha_k e_k \|\} < \infty$, then $x_n = \sum_{k=1}^{n} \alpha_k e_k$ has a weakly convergent subsequence (using reflexivity), $x_{n_i} \overset{\text{wk}}{\to} x$. If $x = \sum_{k=1}^{\infty} \beta_k e_k$, then $\alpha_k = \beta_k$ for all k because $\alpha_k = e_k^*(x_{n_i}) \to e_k^*(x) = \beta_k$. Hence a basis in a reflexive space is boundedly complete.

We now assume that the basis is shrinking and boundedly complete and show that X is reflexive by proving that the unit ball is weakly compact. To this end, let

$$y_n = \sum_{k=1}^{\infty} \beta_k^{(n)} e_k \in \{\|x\| \le 1\}.$$

For a fixed k the sequence $\beta_k^{(n)}$ is bounded, $|\beta_k^{(n)}| \le 2K/\|e_k\|$ (note the for any $x = \sum_{k=1}^{\infty} \alpha_k e_k \in X$ and $n > 0, p \ge 0$ we have $\| \sum_{k=n}^{n+p} \alpha_k e_k \| \le 2K\|x\|$ where K is the basis constant). Hence there are q_n such that for a fixed k, $\beta_k^{(q_n)}$ converges, and, using a diagonal argument, p_n such that $\beta_k^{(p_n)} \to \beta_k$ for some $\beta_k \in \mathbb{R}$. For every n, N we have $\|P_N(y_{p_n})\| \le K\|y_{p_n}\| \le K$ so that $\| \sum_{k=1}^{N} \beta_k e_k \| \le K$ for all N. Because e_k is boundedly complete, there is a $y \in X$ such that $y = \sum_{k=1}^{\infty} \beta_k e_k$.

We claim that y_{p_n} converges weakly to y. Let $f \in X^*$ and $\epsilon > 0$. Since e_k is shrinking, there is an N such that $\sup\{|f(x)| : x = \sum_{k \ge N} \alpha_k e_k, \|x\| = 1\} < \epsilon$. Choose M such that $\| \sum_{k=1}^{N-1} (\beta_k - \beta_k^{(p_n)}) e_k \| < \epsilon$ for $n \ge M$ (remember $\beta_k^{(p_n)} \to \beta_k$ for all k). Recall that $\| \sum_{k=N}^{N+p} \beta_k^{(p_n)} e_k \| \le 2K$ for all $p \ge 0$ and let p, p_n tend to infinity to obtain $\| \sum_{k \ge N} \beta_k e_k \| \le 2K$. Hence for $p_n \ge M$, we have

$$\begin{aligned} |f(y) - f(y_{p_n})| &\le \left| f\left(\sum_{k=1}^{N-1} \left(\beta_k - \beta_k^{(p_n)} \right) e_k \right) \right| + \left| f\left(\sum_{k \ge N} \beta_k e_k \right) \right| + \left| f\left(\sum_{k \ge N} \beta_k^{(p_n)} e_k \right) \right| \\ &\le \|f\|\epsilon + 2K\epsilon + 2K\epsilon, \end{aligned}$$

and $y_{p_n} \overset{\text{wk}}{\to} y$. □

For example, in our favorite non-reflexive spaces c_0 and ℓ^1, the "standard basis" $e_k(l) = \delta_{kl}$ fails to be boundedly complete in c_0 ($\sup_n\{\| \sum_{k=1}^{n} e_k \|_\infty\} < \infty$ but $\sum_{k=1}^{n} e_k$ does not converge in c_0), and fails to be shrinking in ℓ^1 (for instance, if $f(\sum_k \alpha_k e_k) = \sum_k \alpha_k$, then $f(e_k) = 1$ for all k).

We need a few more facts about shrinking bases.

Proposition 3. *If e_k is a shrinking basis for a Banach space $(X, \|\cdot\|)$, then e_k^* is a basis for X^* with coordinate functionals $e_k \in X^{**}$ and the basis constant for e_k^* is no more than the basis constant for e_k. (In fact, e_k^* is a basis for X^* if and only if e_k is shrinking.)*

Proof. First note that for all n we have

$$\left\|\sum_{k=n+1}^{\infty} \alpha_k e_k\right\| \le \left\|\sum_{k=1}^{n} \alpha_k e_k\right\| + \left\|\sum_{k=1}^{\infty} \alpha_k e_k\right\| \le (1+K)\left\|\sum_{k=1}^{\infty} \alpha_k e_k\right\|.$$

If $f \in X^*$, then

$$\begin{aligned} \left|\left(f - \sum_{k=1}^{n} f(e_k) e_k^*\right)\left(\sum_k \alpha_k e_k\right)\right| &= \left|f\left(\sum_{k=n+1}^{\infty} \alpha_k e_k\right)\right| \\ &\le \|f_n\|(1+K)\left\|\sum_k \alpha_k e_k\right\| \to 0 \end{aligned}$$

as $n \to \infty$ because the basis is shrinking, where f_n is the restriction of f to $\overline{\langle e_k : k > n\rangle_{\mathbb{R}}}$. Hence X^* is the closed linear span of the coordinate functionals e_k^*.

For $n \le m$, $\epsilon > 0$, $\alpha_k \in \mathbb{R}$, and $\|\sum_k \beta_k e_k\| = 1$ such that

$$\left|\left(\sum_{k=1}^{n} \alpha_k e_k^*\right)\left(\sum_k \beta_k e_k\right)\right| \ge \left\|\sum_{k=1}^{m} \alpha_k e_k^*\right\| - \epsilon,$$

we have

$$\begin{aligned} \left\|\sum_{k=1}^{n} \alpha_k e_k^*\right\| &\le \left|\left(\sum_{k=1}^{n} \alpha_k e_k^*\right)\left(\sum_k \beta_k e_k\right)\right| + \epsilon \\ &= \left|\left(\sum_{k=1}^{m} \alpha_k e_k^*\right)\left(\sum_{k=1}^{n} \beta_k e_k\right)\right| + \epsilon \\ &\le \left\|\sum_{k=1}^{m} \alpha_k e_k^*\right\|\left\|\sum_{k=1}^{n} \beta_k e_k\right\| + \epsilon \\ &\le K\left\|\sum_{k=1}^{m} \alpha_k e_k^*\right\|\left\|\sum_k \beta_k e_k\right\| + \epsilon \\ &= K\left\|\sum_{k=1}^{m} \alpha_k e_k^*\right\| + \epsilon, \end{aligned}$$

where K is the basis constant for e_k. Hence e_k^* is a basis of X^*, with basis constant smaller than K. Finally, it is clear that the e_k act as the coordinate functionals for the e_k^*. $\square$

Finally, we characterize X^{**} for Banach spaces with a shrinking basis.

Proposition 4. *Let $(X, \|\cdot\|)$ be a Banach space with shrinking basis e_k,*

$$Z = \left\{ \{\alpha_k\}_k \in \mathbb{R}^{\mathbb{N}} : \sup_n \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \right\} < \infty \right\}, \text{ and } \|\{\alpha_k\}_k\|_Z = \sup_n \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \right\}.$$

Then $(Z, \|\cdot\|_Z)$ is a Banach space and the map $T : X^{**} \to Z$ given by $T(\phi) = \{\phi(e_k^*)\}_k$ is an isomorphism, isometric if e_k is monotone.

Proof. The proof that Z is a Banach space is similar to that of Lemma 1, and T is clearly linear. If $T(\phi) = \{\phi(e_k^*)\}_k = 0$, then for every $f \in X^*$ we have

$$\phi(f) = \phi\left(\sum_k f(e_k)e_k^*\right) = \sum_k f(e_k)\phi(e_k^*) = 0,$$

so that T is injective. To see that T is surjective, let $\{\alpha_k\}_k \in Z$, $f \in X^*$, and $n \le m$. We have

$$\left| \sum_{k=n}^m \alpha_k f(e_k) \right| \le \|f_n\| \left\| \sum_{k=n}^m \alpha_k e_k \right\| \le (1+K)\|f_n\| \|\{\alpha_k\}_k\|_Z \to 0$$

as $n \to \infty$ because e_k is shrinking (here f_n is the restriction of f to $\overline{\langle e_k : k > n \rangle_{\mathbb{R}}}$). Hence $\sum_k \alpha_k f(e_k)$ converges and we can define $\phi_0 : X^* \to \mathbb{R}$ by $\phi_0(f) = \sum_k \alpha_k f(e_k)$, which is clearly linear with $T(\phi_0) = \{\alpha_k\}_k$, and bounded because for any n

$$\left| \sum_{k=1}^n \alpha_k f(e_k) \right| \le \|f\| \left\| \sum_{k=1}^n \alpha_k e_k \right\| \le K \|f\| \|\{\alpha_k\}_k\|_Z.$$

Finally, we have the bounded

$$\begin{aligned} \|T(\phi)\|_Z &= \sup_n \left\{ \left\| \sum_{k=1}^n \phi(e_k^*) e_k \right\| \right\} = \sup_{n, \|f\|=1} \left\{ \left\| \sum_{k=1}^n \phi(e_k^*) f(e_k) \right\| \right\} \\ &= \sup_{n, \|f\|=1} \left\{ \left\| \phi\left(\sum_{k=1}^n f(e_k) e_k^* \right) \right\| \right\} \ge \sup_n \{\phi(f)\} \\ &= \|\phi\|, \end{aligned}$$

since $\sum_{k=1}^n f(e_k)e_k^* \to f$. So if $K = 1$, T is an isometry. □

The last two propositions show that if X has a shrinking basis, we can identify X^* and X^{**} with sequence spaces.

2 James' space

We now introduce the subject of this paper, the space

$$J = \{x \in c_0(\mathbb{R}) : \|x\|_J < \infty\}$$

where $\|\cdot\|_J$ is defined by

$$\|x\|_J = \sup \left\{ \left(\sum_{k=1}^l |x(p_{k+1}) - x(p_k)|^2 \right)^{1/2} : \ 2 \le l, 1 \le p_1 < \cdots < p_l \right\}.$$

We will also make use of the equivalent norm (for which the isometric isomorphism $J \cong J^{**}$ holds) given by

$$\|x\|_{cyc} = \sup\left\{ \left(|x(p_l) - x(p_1)|^2 + \sum_{k=1}^{l} |x(p_{k+1}) - x(p_k)|^2 \right)^{1/2} \ : \ 2 \leq l, 1 \leq p_1 < \cdots < p_l \right\}.$$

The space J has some interesting properties, such as

1. J is not reflexive and does not contain any subspaces isomorphic to c_0 or ℓ^1,
2. J has codimension one in J^{**} under the canonical injection $\iota : J \to J^{**}$,
3. J (with the "cyclic quadratic variation" norm above) is isometrically isomorphic to J^{**} (but not via the canonical injection!),
4. J is not the underlying real space of a complex Banach space (we do not show this, but it follows from the fact that $\dim_{\mathbb{R}}(J^{**}/J) = 1$ is odd since if X is the underlying real space of the complex space X_0, then $X^{**}/X \cong X_0^{**}/X_0$, real duals on the left, complex duals on the right),
5. J does not have an unconditional basis (we do not show this, but a Banach space with an unconditional basis is either reflexive, contains c_0, or contains ℓ^1).

Properties 1,2, and 3 follow from the next series of propositions.

Proposition 5. *$(J, \|\cdot\|_J)$ is a Banach space and e_k defined by $e_k(l) = \delta_{kl}$, is a monotone basis for J*

Proof. That $\|\cdot\|_J$ is a norm on J is obvious and we now show completeness. If $x_j \in J$ is such that $\sum_j \|x_j\|_J < \infty$, then $|x_j(k)| = \lim_{n\to\infty} |x_j(k) - x_j(n)| \leq \|x_j\|_J$, $\sum_{j=1}^{\infty} x_j(k) =: \alpha_k$ exists and $\alpha_k \to 0$ (since c_0 is complete). For $x = \sum_{k=1}^{\infty} \alpha_k e_k$ we have

$$\begin{aligned}
\|x\|_J &= \sup\left\{ \left(\sum_{k=1}^{l} |x(p_{k+1}) - x(p_k)|^2 \right)^{1/2} \ : \ 2 \leq l, 1 \leq p_1 < \cdots < p_l \right\} \\
&\leq \sup\left\{ \sum_{j=1}^{\infty} \left(\sum_{k=1}^{l} |x_j(p_{k+1}) - x_j(p_k))|^2 \right)^{1/2} \ : \ 2 \leq l, 1 \leq p_1 < \cdots < p_l \right\} \\
&\leq \sum_j \|x_j\|_J < \infty.
\end{aligned}$$

To see that e_k is monotone, let $p \leq q$ and note that

$$\begin{aligned}
\left\| \sum_{k=1}^{p} \alpha_k e_k \right\|_J &= \sup\left\{ \left(\sum_{k=1}^{l} |\alpha_{p_{k+1}} - \alpha_{p_k}|^2 \right)^{1/2} \ : \ 2 \leq l, 1 \leq p_1 < \cdots < p_l \leq p \right\} \\
&\leq \sup\left\{ \left(\sum_{k=1}^{l} |x(p_{k+1}) - x(p_k)|^2 \right)^{1/2} \ : \ 2 \leq l, 1 \leq p_1 < \cdots < p_l \leq q \right\} \\
&= \left\| \sum_{k=1}^{q} \alpha_k e_k \right\|_J
\end{aligned}$$

(with equality when extending by zero for instance).

Finally, to see that $J = \overline{\langle e_k \rangle_{\mathbb{R}}}$, let $x \in X$, $\epsilon > 0, 2 \leq l, 1 \leq p_1 < \cdots < p_l$ such that

$$\left(\sum_{k=1}^{l} |x(p_{k+1}) - x(p_k)|^2 \right)^{1/2} \geq \|x\|_J - \epsilon.$$

Then $\|x - \sum_{k=1}^{l} x(k) e_k\|_J < \epsilon$, and e_k is a basis for J.

□

Proposition 6. *$(J, \|\cdot\|_J)$ is not reflexive (more specifically, e_k is not boundedly complete).*

Proof. Let $s_n = \sum_{k=1}^{n} e_k$. Then $\|s_n\|_J = 1$ for every $n \geq 1$, but s_n does not converge in J, e.g. $\|s_m - s_n\|_J = \sqrt{2}$ for $n \neq m$. □

Proposition 7. *The basis e_k is shrinking.*

Proof. If not, there is an $f \in X^*, \epsilon > 0$, and $y_i = \sum_{k=n_i}^{n_{i+1}} \beta_k^{(i)} e_k$ with n_i increasing to infinity, $\|y_i\|_J = 1$, and $f(y_i) > \epsilon$ for all i. If $y = \sum_i y_i \in J$, then $f(y) > \epsilon \sum_i 1/i = \infty$, a contradiction. However $y \in J$. Let $\epsilon > 0$ and $2 \leq l, 1 \leq p_1 < \cdots < p_l$ such that

$$\|y\|_J^2 - \epsilon \leq \sum_{k=1}^{l} |y(p_{k+1}) - y(p_k)|^2.$$

Each term in the sum is either of the form

$$\frac{1}{i^2}(y_i(p_{k+1}) - y_i(p_k))^2 \text{ or } \left(\frac{y_{i+j}(p_{k+1})}{i+j} - \frac{y_i(p_k)}{i} \right)^2 \leq 2 \left(\frac{y_{i+j}(p_{k+1})}{i+j} \right)^2 + 2 \left(\frac{y_i(p_k)}{i} \right)^2$$

depending on whether or not we are in the same "block". We get

$$\|y\|_J^2 \leq 5 \sum_{i=1}^{\infty} \frac{\|y_i\|_J^2}{i^2} < \infty$$

and $y \in J$. □

Proposition 8. *$J^{**} = \iota(J) \oplus \mathbb{R} s_\infty$, where s_∞ is the weak-$*$ limit of $s_n = \sum_{k=1}^{n} e_k$.*

Proof. From Proposition 4, J^{**} is isomorphic the the space Z of seqences $\{\alpha_k\}_k$ such that $\|\{\alpha_k\}_k\|_Z = \sup_n \{\| \sum_{k=1}^{n} \alpha_k e_k \|_J\} < \infty$. If $\{\alpha_k\}_k$ is such a sequence, then $\lambda = \lim_k \alpha_k$ clearly exists, and $\sum_k \alpha_k e_k \in J$ if and only if $\lambda = 0$. Hence any $\phi = \{\alpha_k\}_k \in J^{**}$ can be decomposed as

$$\phi = \{\alpha_k - \lambda\}_k + \lambda s_\infty.$$

□

Proposition 9. *J is isometrically isomorphic to J^{**} under $\|\cdot\|_{cyc}$.*

Proof. Define $U : Z \to (J, \|\cdot\|_{cyc})$ (with Z isometrically isomorphic to J^{**} as in Proposition 4) by

$$U(\{\alpha_k\}_k) = -\lambda e_1 + \sum_{k>1} (\alpha_k - \lambda) e_k$$

where $\lambda = \lim_k \alpha_k$ as in the previous proposition. U is clearly linear, and surjective since for $\sum_k \alpha_k e_k \in J$ we have

$$U((\alpha_2, \alpha_3, \dots) - \alpha_1 s_\infty) = \sum_k \alpha_k e_k,$$

with s_∞ as in the previous proposition. To see that U is an isometry, let $\{\alpha_k\}_k \in Z$ and compute

$$\begin{aligned}
&\|U(\{\alpha_k\}_k)\|_{cyc}^2 \\
&= \sup\left\{\max\left\{(\alpha_{p_l} - \alpha_{p_1})^2 + \sum_{k=1}^{l}(\alpha_{p_{k+1}} - \alpha_{p_k})^2, \alpha_{p_l}^2 + \alpha_{p_1}^2 + \sum_{k=1}^{l-1}(\alpha_{p_{k+1}} - \alpha_{p_k})^2\right\} : l, p_k\right\}
\end{aligned}$$

with the "max" term coming from whether or not the initial $-\lambda$ term is used to compute the cyclic quadratic variation. On the other hand we have

$$\begin{aligned}
\|\{\alpha_k\}_k\|_Z^2 &= \sup_n\left\{\left\|\sum_{k=1}^{n}\alpha_k e_k\right\|_{cyc}\right\} \\
&= \sup\left\{\max\left\{(\alpha_{p_l} - \alpha_{p_1})^2 + \sum_{k=1}^{l}(\alpha_{p_{k+1}} - \alpha_{p_k})^2, \alpha_{p_l}^2 + \alpha_{p_1}^2 + \sum_{k=1}^{l-1}(\alpha_{p_{k+1}} - \alpha_{p_k})^2\right\} : l, p_k\right\}
\end{aligned}$$

with the "max" term coming from whether or not any of the "trailing zeros" were used in the computation of the cyclic quadratic variation. Therefore U is an isometry. □

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