The Banach-Tarski Paradox

Robert Hines

May 3, 2017

Abstract

We give a proof of "doubling the ball" using non-amenability of the free group on two generators, which we show is a subgroup of SO_3.

Taking the five loaves and the two fish and looking up to heaven, he gave thanks and broke the loaves. Then he gave them to the disciples, and the disciples gave them to the people. They all ate and were satisfied, and the disciples picked up twelve baskets full of broken pieces that were left over.

1 Introduction

Theorem 1 (Banach-Tarski)**.** *Any two bounded subsets $A, B \subseteq \mathbb{R}^n$, $n \geq 3$, with non-empty interior are piecewise congruent (there are finite partitions $A = \coprod_i A_i$, $B = \coprod_i B_i$ such that $A_i = g_i(B_i)$ for some $g_i \in \mathbb{R}^n \rtimes SO(3) \cong Isom^+(\mathbb{R}^n)$).*

In particular, this shows that there is no finitely additive isometry invariant measure defined on all subsets of $\mathbb{R}^3$.

2 Warm-up: Vitali set

Here is the classic example of a non-measureable set (which uses the axiom of choice of course). Let T be a set of coset representatives for $S^1/e^{2\pi i\mathbb{Q}}$ (i.e. *choose* a transversal). Then $S^1 = \coprod_{q\in\mathbb{Q}\cap[0,1)} T_q$ where $T_q = e^{2\pi iq}T$, and $\mu(T_q) = \mu(T_0)$ since the T_q are all isometric. If $\mu(T_0) = 0$ then $\mu(S^1) = 0$ and if $\mu(T_0) > 0$ then $\mu(S^1) = \infty$.

3 Free Groups

Definition 1. *The* ***free group on the set*** *X, $X \subseteq F(X)$, is a group satisfying the following universal property: given a group G and a function $f : X \to G$, there is a unique homomorphism $F(X) \to G$ extending f.*

A construction is given as follows. Let W be the set of all words in the letters $X \cup X^{-1}$ and let $F(X)$ be the set of equivalence classes of W under the equivalence of insertion/deletion of a subword of the form xx^{-1} or $x^{-1}x$. Then $F(X)$ is a group under concatenation with identity the empty word.

4 Ping-Pong Lemma

Lemma 1. *If $g, h \in Sym(X)$, $A, B \subseteq X$, $A\Delta B \neq \emptyset$, and*

$$g^n(A) \subseteq B,\ h^n(B) \subseteq A,\ 0 \neq n \in \mathbb{Z},$$

then the subgroup generated by g, h is free on the generators g, h.

Proof. Let w be a non-empty reduced word in g, h, g^{-1}, h^{-1}. We want to show that $w \neq 1$. Without loss of generality, assume that

$$w = g^{n_1}h^{m_1}\dots g^{n_{k-1}}h^{m_{k-1}}g^{n_k},$$

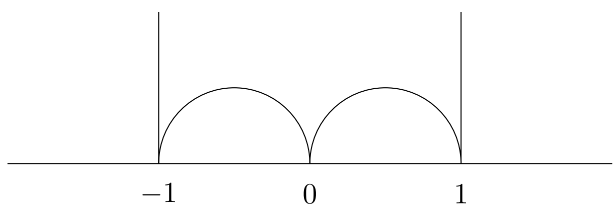
(else conjugate by an appropriate power of g). We have

$$w(A) \subseteq g^{n_1}h^{m_1}\dots g^{n_{k-1}}h^{m_{k-1}}g^{n_k}(A) \subseteq g^{n_1}h^{m_1}\dots g^{n_{k-1}}h^{m_{k-1}}(B) \subseteq \dots \subseteq g^{n_1}(A) \subseteq B.$$

Hence w is not the identity (else $w(A) = A \not\subseteq B$). $\square$

Two examples:

- The group generated by $g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ acts bijectively on $P^1(\mathbb{R})$ as fractional linear transformations, $g = x + 2$, $h = \frac{1}{2x+1}$. Letting $A = [-1, 1]$, $B = [-\infty, -1] \cup [1, \infty]$, we have $g^n(A) \subseteq B$, $h^n(B) \subseteq A$ (g^n translates A into B, and h^n shrinks B to the interval$\left[\frac{1}{1-2n}, \frac{1}{1+2n}\right]$).



- Almost every (with respect to Haar measure) pair of elements from $SO(3)$ generates a free group. To see this, note that every word W in the free group defines a map $W : SO(3) \times SO(3) \to SO(3)$ and that $W^{-1}(1)$ is either $SO(3) \times SO(3)$ or has measure zero. However, no relation holds since $SO(3)$ contains a free group on two generators. Explicitly, if $\cos\theta$ is transcendental, then

$$\psi = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \phi = \begin{pmatrix} -\cos\theta & 0 & -\sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{pmatrix},$$

generate a group isomorphic to $PSL_2(\mathbb{Z})$, $\langle \psi, \phi \mid \psi^3 = \phi^2 = 1 \rangle$ (cf. [3]). This will contain many free groups, such as the previous example. Or, if $\theta = \arccos(1/3)$, then

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

generatre a free group (cf. [1]). To prove this, show by induction that any reduced word W in $\{A, A^{-1}, B, B^{-1}\}$ ending in A has first column $\frac{1}{3^m}(a, b\sqrt{2}, c)$ where m is the length of W, and a, b, c are integers with $3 \nmid b$ (in particular W is not the identity).

Finitely generated subgroups of non-solvable Lie groups are almost always free, basically using the argument above (see [2]).

5 Paradoxical Sets

Definition 2. *Let $G \leq Sym(X)$. Two subsets $A, B \subseteq X$ are G-**congruent** if there is a $g \in G$ such that $g(A) = B$. Two subsetes A, B are **piecewise** G-**congruent** if $A = \coprod_i A_i$, $B = \coprod_i B_i$ and there are $g_i \in G$ such that $g_i(A_i) = B_i$. The subset $E \subseteq X$ is G-**paradoxical** if there is a partition $E = A \coprod B$ such that both A and B are piecewise G-congruent to E.*

Lemma 2. *The free group on two generators $F_2 = \langle a, b \rangle$ is F_2-paradoxical (with respect to the left action on itself).*

Proof. We have

$$F_2 = \{1\} \cup W_a \cup W_{a^{-1}} \cup W_b \cup W_{b^{-1}}$$

where W_x is the collection of reduced words starting with x. We also have $F_2 = aW_{a^{-1}} \cup W_a = bW_{b^{-1}} \cup W_b$, almost giving a paradoxical decomposition. Modify W_a, $W_{a^{-1}}$ as follows

$$W'_a = W_a \setminus \{a^n : n \geq 1\},\ W'_{a^{-1}} = W'_{a^{-1}} \cup \{a^n : n \geq 0\}.$$

Then

$$F_2 = W'_a \cup W'_{a^{-1}} \cup W_b \cup W_{b^{-1}}$$

and

$$F_2 = bW_{b^{-1}} \cup W_b \sim W_b \cup W_{b^{-1}},\ F_2 = aW'_{a^{-1}} \cup W'_a \sim W'_a \cup W'_{a^{-1}}$$

where $\sim$ indicates piecewise F_2-congruence. (Draw a picture of the Cayley graph.) □

6 Doubling the 3-ball

Here we prove the following (weaker) version of the Banach-Tarski paradox.

Theorem 2. *The unit ball D^3 is piecewise $SO(3)$-congruent to $D^3 \coprod D^3$.*

The heart of the proof is the fact that $SO(3)$ contains a free group on two generators as discussed above. We proceed in several steps.

- [Step 1] We start by showing that S^2 is piecewise congruent to $S^2 \setminus C$ for any countable set $C \subseteq S^2$. Let l be a line through the origin such that $S^2 \cap l \cap C = \emptyset$, A be the set of angles α such that ρ_α, rotation by α about l, takes a point of C to C, and A' the union $\cup_{n \geq 1} \frac{1}{n} A$. Then A' is countable and for any angle $\theta \notin A'$ and any $n \geq 1$, we have $\rho_\theta^n(C) \cap C = \emptyset$. Finally, if $\mathcal{O} = \cup_{n \geq 0} \rho_\theta(C)$, then

$$S^2 \setminus C = (S^2 \setminus \mathcal{O}) \coprod (\mathcal{O} \setminus C),\ \mathcal{O} \setminus C = \rho_\theta(\mathcal{O}),$$

so that $S^2 = (S^2 \setminus \mathcal{O}) \coprod \mathcal{O}$ is piecewise congruent to $S^2 \setminus C$.

- [Step 2] Next, we want to show that S^2 is piecewise congruent to $S^2 \coprod S^2$. We know that there is a copy of F_2 in $SO(3)$. Let C be the set of intersections of the axes of ρ_w, $w \in F_2$, with S^2 (here ρ_w is the rotation correspond to the word $w \in F_2$). Then S^2 is piecewise congruent to $S^2 \setminus C$ by the above, and $S^2 \setminus C$ is a disjoint union of F_2-orbits. Let D be a **choice** of an element from each F_2-orbit of $S^2 \setminus C$. Then $F_2 D = S^2 \setminus C$ and the action of F_2 is free (C was exactly the set of fixed points of the action of F_2 on the sphere). We get a partition

$$S^2 \setminus C = W'_a D \cup W'_{a^{-1}} D \cup W_b D \cup W_{b^{-1}} D$$

induced by the partition of F_2. Hence $S^2 \setminus C$ is piecewise F_2-congruent to two copies of itself, and each of those is piecewise congruent to S^2.

- [Step 3] Step 2 shows that $D^3 \setminus \{0\}$ is piecewise congruent to two copies of itself (extending radially), so we want to show that D^3 is piecewise congruent to $D^3 \setminus \{0\}$. We have

  $$D^3 = (D^3 \setminus S) \coprod S, \ S \sim S \setminus \{0\},$$

  where S is a sphere of radius $1/2$ passing through the origin. Hence $D^3 \sim D^3 \setminus \{0\}$ as desired.

References

[1] Cornelia Drutu, *Geometry of Infinite Groups*, `https://people.maths.ox.ac.uk/drutu/gradLN08.pdf`

[2] D. B. A. Epstein, *Almost all subgroups of a Lie group are free*, Journal of algebra 19, 261–262, 1971.

[3] Karl Stromberg, *The Banach-Tarski Paradox*, American Mathematical Monthly, Vol. 86, No. 3 (Mar. 1979), pp. 151-161.

[4] Grzegorz Tomkowicz and Stan Wagon, *The Banach Tarski Paradox* Second Edition, Encyclopedia of Mathematics and its Applications 163, Cambridge University Press, 2016