The Gelfand-Naimark-Segal construction

Robert Hines

December 22, 2017

Introduction

Our goal is to show that every C^*-algebra is isometrically $*$-isomorphic to a $*$-subalgebra of bounded linear operators on some Hilbert space. This is accomplished via the Gelfand-Naimark-Segal construction ([GN], [S]) which uses a positive linear functional f and the left regular representation $A \to \mathrm{End}_{\mathbb{C}}(A)$ to produce a cyclic (irreducible if f is a "pure state") representation $\pi_f : A \to \mathcal{B}(\mathcal{H})$, along with the fact that there are "enough" positive linear functionals. Sources for the exposition below are [C], [A], and [D].

Preliminaries

A **C^*-algebra** A is a unital Banach algebra (a complete normed linear space over $\mathbb{C}$ with continuous associative multiplication $A \times A \to A$ and $\|1\| = 1$) with an map $* : A \to A$ satisfying

$$(a^*)^* = a,\ (\alpha a + b)^* = \bar{\alpha} a^* + b^*,\ (ab)^* = b^* a^*,\ \|a^* a\| = \|a\|^2,\ \alpha \in \mathbb{C},\ a, b \in A.$$

We will denote by A' the dual space of A, $A' := \mathcal{L}(A, \mathbb{C})$ (bounded linear functionals).

Examples:

- If X is a compact Hausdorff topological space and $A = C(X)$ is the ring of complex-valued continuous functions on X, then A is a commutative C^*-algebra with involution $f^* = \bar{f}$ (point-wise complex conjugation of function values).
- If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$, the bounded linear endomorphisms of $\mathcal{H}$, is a C^*-algebra with $*$ the usual adjoint defined by $\langle Tx, y \rangle = \langle x, T^* y \rangle$.

A **representation** of a C^*-algebra is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. Two representations $\pi_1 : A \to \mathcal{B}(\mathcal{H}_1)$, $\pi_2 : A \to \mathcal{B}(\mathcal{H}_2)$, are (unitarily) **equivalent** if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U \pi_1(a) = \pi_2(a) U,\ a \in A.$$

A representation π is **cyclic** if there exists $e \in \mathcal{H}$ such that $\{\pi(a)e : a \in A\}$ is dense in $\mathcal{H}$. A representation is (topologically) **irreducible** if it has no proper, nontrivial closed invariant subspaces, i.e. if $V \subseteq \mathcal{H}$ is closed and $\pi(A)V \subseteq V$, then $V \in \{0, \mathcal{H}\}$. Equivalently (see Appendix), the commutant of $\pi(A)$ is $\mathbb{C}$, where the **commutant** of a collection of operators $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is

$$\{T \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } S \in \mathcal{S}\}.$$

An application of Zorn's lemma shows that every non-degenerate representation ($\{\pi(a)v : a \in A, v \in \mathcal{H}\}$ is dense in $\mathcal{H}$) is equivalent to an orthogonal direct sum of cyclic representations.

We conclude this section by recalling the functional calculus for normal elements of a C^*-algebra. For commutative C^*-algebras we have the following.

Theorem 1. *If A is a commutative C^*-algebra and Σ is the space of maximal ideals of A (equivalently the collection of homomorphisms $A \to \mathbb{C}$ with the weak* topology), then the Gelfand transform*

$$\Gamma : A \to C(\Sigma),\ \Gamma(a)x = x(a),$$

is an isometric $*$-isomorphism.

For a commutative C^*-algebra A generated by a normal element a (i.e. a commutes with its adjoint a^*), we can naturally identify the maximal ideal space Σ with the the spectrum of a, $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^\times\}$.

Proposition 1. *If A is a commutative C^*-algebra with generator a and maximal ideal space Σ, then*

$$\tau : \Sigma \to \sigma(a),\ \tau(x) = x(a),$$

is a homeomorphism. Moreover, if $p(z, \bar{z})$ is a polynomial in z and $\bar{z}$, then

$$\Gamma(p(a, a^*)) = p \circ \tau$$

where Γ is the Gelfand transform.

Recall that if $A \subseteq B$ are C^*-algebras, then $\sigma_A(a) = \sigma_B(a)$, i.e. the spectrum of an element does not depend on the algebra. We can now define the **functional calculus** for normal elements of a C^*-algebra.

Theorem 2. *Let A be a C^*-algebra, $a \in A$ normal, and Γ, τ as above. Define $\rho : C(\sigma(a)) \to C^*(a) \subseteq A$, where $C^*(a)$ is the $*$-subalgebra generated by a, by the following commutative diagram*

$$\begin{tikzcd} C^*(a) \arrow[rr, "\Gamma"] & & C(\Sigma) \\ & C(\sigma(a)) \arrow[ul, "\rho"] \arrow[ur, "\tau*"'] & \end{tikzcd}$$

where $\Sigma \cong \sigma(a)$ is the maximal ideal space of $C^*(a)$. The map $\rho : C(\sigma(a)) \to A$ is the functional calculus for a, taking functions on $\sigma(a)$ to elements of A, which we think of as evaluation as in the previous proposition. The functional calculus $\rho : C(\sigma(a)) \to A$ is an isometric $*$-homomorphism.

Positive elements and positive functionals

Denote by $A_+ = \{a^*a : a \in A\}$ the **positive** elements of A. Equivalently (see appendix), $a \in A_+$ if and only if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$. A linear functional $f : A \to \mathbb{C}$ is **positive** if $f(a^*a) \geq 0$, i.e. $f(A_+) \subseteq \mathbb{R}_{\geq 0}$. Here are two examples:

- The continuous linear functionals on $C(X)$, X compact, are the finite complex Borel measures and μ is positive if $\int |\phi|^2 d\mu \geq 0$ for all $\phi \in C(X)$, i.e. the positive measures.

- If $a \in \mathcal{B}(\mathcal{H})$ and $v \in \mathcal{H}$, then

$$f(a) := \langle av, v \rangle$$

is a positive linear functional on $\mathcal{B}(\mathcal{H})$ since

$$f(a^*a) = \langle a^*av, v \rangle = \langle av, av \rangle = \|av\|^2 \geq 0.$$

As a sort of converse to the second example, given a positive functional $f : A \to \mathbb{C}$, we have a positive semi-definite Hermitian form $A \times A \to A$ defined by $f(y^*x)$. A positive linear functional f is automatically continuous, $\|f\| = f(1)$, as follows. By Cauchy-Schwarz we have

$$|f(x)|^2 \leq f(1)f(x^*x)$$

so we must show that $f(x^*x) \leq f(1)$ if $\|x\| \leq 1$, i.e. $f(1 - x^*x) \geq 0$ for $\|x\| \leq 1$. For any self-adjoint a with $\|a\| \leq 1$, an application of the functional calculus shows that $1 - a$ is positive, $\sigma(1 - a) \subseteq [0, 2]$.

The positive linear functionals of norm one are known as the **states** of A, denoted by S_A. The states of A form a compact covex subset of A' in the weak* topology; S_A is clearly convex and closed

$$S_A = \bigcap_{a \in A_+} \{f \in A' : f(a) \in [0, \infty)\},$$

hence compact by the Banach-Alaoglu theorem. By Krein-Milman, there are extreme points of S_A, so-called **pure states**, whose importance will be discussed later. To end this section, we identify states as the linear functionals satisfying $\|f\| = f(1) = 1$.

Lemma 1 ([A], Corollary to Theorem 1.7.1 or [D], Proposition 2.1.9)**.** *If $f \in A'$ satisfies $1 = f(1) = \|f\|$, then f is a state.*

Proof. What we will actually show is that for any such f and for any normal element $a \in A$, $f(a)$ is in the closed convex hull K of $\sigma(a)$. First note that K is the intersection of all closed disks containing $\sigma(a)$ (this is true for any compact convex set in the plane). So if $f(a) \notin K$, then there is an $R > 0$ and $z_0 \in \mathbb{C}$ such that $\sigma(a) \subseteq \{z : |z - z_0| \leq R\}$ but $|f(a) - z_0| > R$. In particular the spectral radius of $a - z_0$, $r(a - z_0)$, is less or equal R. By the functional calculus, $r(a - z_0) = \|a - z_0\|$. However, $|f(a - z_0)| > R$ gives a contradiction (using $\|f\| = f(1) = 1$). □

GNS

The following construction of representations is known as the GNS construction after Gelfand, Naimark, and Segal ([GN], [S]). The basic idea is to use a positive linear functional to turn a quotient of the left regular representation of a C^*-algebra into a representation.

Theorem 3 ([C], Theorem VIII.5.14)**.** *Given a positive functional $f : A \to \mathbb{C}$, there is a cyclic representation $\pi_f : A \to \mathcal{B}(\mathcal{H}_f)$ with generator $e \in \mathcal{H}_f$ such that*

$$\langle \pi_f(a)e, e \rangle = f(a).$$

If $\pi : A \to \mathcal{H}$ is another cyclic representation with generator e' such that $f(a) = \langle \pi(a)e', e' \rangle$, then π and π_f are unitarily equivalent.

Proof. For a positive linear functional f, let $N = \{a \in A : f(a^*a) = 0\}$. This is clearly a closed subspace, and in fact is a left ideal: if $n \in N$, $a \in A$, then

$$f((an)^*(an))^2 = f(n^*(a^*an))^2 \leq f(n^*n)f((a^*an)^*(a^*an)) = 0$$

by Cauchy-Schwarz. Consider the quotient A/N as a vector space, and define the inner product

$$\langle x + N, y + N\rangle_f = f(y^*x).$$

This is well-defined since if $x, y \in A$, $a, b \in N$, then

$$f((y + b)^*(x + a)) = f(y^*x) + f(y^*a) + f(b^*x) + f(b^*a)$$

and each of the last three terms on the right is zero, e.g.

$$|f(b^*a)|^2 \leq f(a^*a)f(b^*b) = 0.$$

Let $\mathcal{H}_f$ be the completion of A/N with respect to the norm $\|x + N\|_f = \sqrt{\langle x + N, x + N\rangle_f}$. The left regular representation,

$$A \to \mathrm{End}_{\mathbb{C}}(A),\ a \mapsto L_a,\ L_a(x) = ax,$$

gives a representation

$$\pi : A \to \mathrm{End}_{\mathbb{C}}(A/N),\ a \mapsto \widetilde{L_a},\ \widetilde{L_a}(x + N) = ax + N,$$

because N is a left ideal (here we mean 'representation' in the purely algebraic sense). We want to show that π is continuous (with respect to $\|\cdot\|_f$) and extends to a representation $\pi_f : A \to \mathcal{B}(\mathcal{H}_f)$ of C^*-algebras.

First we show that each $\pi(a)$ is bounded, $\|\pi(a)(x+N)\| \leq \|x+N\|_f\|a\|$. We have $\|\pi(a)(x+N)\|_f^2 = f(x^*a^*ax)$ so we want to show that $f(x^*a^*ax) \leq f(x^*x)\|a\|^2$. The functional $g(b) = f(x^*bx)$ is positive, with norm $\|g\| = g(1) = f(x^*x)$ so that

$$f(x^*a^*ax) = g(a^*a) \leq \|g\|\|a^*a\| = f(x^*x)\|a\|^2.$$

Hence $\pi(a)$ extends to $\pi_f(a) \in \mathcal{B}(\mathcal{H}_f)$. The homomorphism $\pi_f : A \to \mathcal{B}(\mathcal{H}_f)$ is clearly continuous, $\|\pi_f\| \leq 1$. The representation π_f is cyclic since $\pi_f(A)(1 + N)$ is dense in H_f.

Now let $\pi : A \to \mathcal{H}$ be another representation with generator $e' \in \mathcal{H}$ satisfying

$$\langle \pi(a)e', e'\rangle = f(a) = \langle \pi_f(a)e, e\rangle_f.$$

Define U on the dense subspace $\pi_f(A)e_f \subseteq \mathcal{H}_f$ by $U\pi_f(a)e_f = \pi(a)e'$, a well-defined isometry since

$$\|\pi(a)e'\|^2 = \langle \pi(a)e', \pi(a)e'\rangle = \langle \pi(a^*a)e', e'\rangle = \langle \pi_f(a^*a)e, e\rangle = \|\pi_f(a)e\|_f^2.$$

This extends to all of $\mathcal{H}_f$ and satisfies

$$U\pi_f(a)\pi_f(b)e_f = U\pi_f(ab)e_f = \pi(ab)e = \pi(a)\pi(b)e = \pi(a)U\pi_f(b),$$

so that $U\pi_f(a) = \pi(a)U$ and U intertwines π_f and π. $\square$

For a positive linear functional f and a constant $\alpha > 0$, π_f and $\pi_{\alpha f}$ are unitarily equivalent via

$$U : \mathcal{H}_f \to \mathcal{H}_{\alpha f},\ Ux = x/\sqrt{\alpha},$$

so we need only consider the positive functionals of norm one, S_A. Among the states, the pure states correspond to irreducible representations.

Proposition 2 ([A], Theorem 1.6.6)**.** *The pure states of A correspond to the irreducible representations of A, i.e. if π is a cyclic representation with generator e, $\|e\| = 1$, and $f(a) = \langle \pi(a)e, e\rangle$, then f is pure if and only if π is irreducible.*

Proof. Assume f is a pure state, and suppose $E \neq 0, 1$ is a projection in the commutant of $\pi(A)$, with $t := \|Ee\|^2 < 1$. Define linear functionals $g_i \in A'$ as follows

$$g_1(a) = \frac{1}{t}\langle \pi(a)e, Ee\rangle,\ g_2(a) = \frac{1}{1-t}\langle \pi(a)e, E^{\perp}e\rangle,$$

where $E^{\perp} = 1 - E$. Clearly $f = g_1 + g_2$ and the g_i are states since

$$tg_1(a) = \langle \pi(a)e, Ee\rangle = \langle \pi(a)e, E^2e\rangle = \langle E\pi(a)e, Ee\rangle = \langle \pi(a)Ee, Ee\rangle$$

shows that g_1 is positive and $g_1(1) = \|Ee\|^2/t = 1$, so that g_1 is a state (by Lemma 1 above). A similar argument shows that g_2 is a state as well. Since f is pure, we have $f = g_1$, i.e. $\langle \pi(a)e, Ee\rangle = t\langle \pi(a)e, e\rangle$ for all $a \in A$. Hence $E - t = 0$ (by density of $\pi(A)e$), a contradiction.

Conversely, suppose π is irreducible and that $f = tg_1 + (1-t)g_2$ for some $g_1, g_2 \in S_A$. We will show that $g_1 = f$. Define a (positive semi-definite) sesquilinear form on the dense subspace $\pi(A)e \subseteq \mathcal{H}$ by

$$[\pi(a)e, \pi(b)e] := tg_1(b^*a)$$

We have

$$0 \leq tg_1(a^*a) = f(a^*a) - (1-t)g_2(a^*a) \leq f(a^*a)$$

so that $[\cdot,\cdot]$ is bounded. By the Riesz lemma, there is some T such that $[x, y] = \langle x, Ty\rangle$. T commutes with $\pi(A)$ since equality holds below:

$$tg_1((bc)^*a) = \langle \pi(a)e, H\pi(b)\pi(c)e\rangle \stackrel{?}{=} \langle \pi(a)e, \pi(b)H\pi(c)e\rangle = \langle \pi(b^*a)e, H\pi(c)e\rangle = tg_1(c^*b^*a).$$

Since π is irreducible, $H = r \cdot 1$ is a scalar and we have

$$tg_1(a) = tg_1(1^*a) = \langle \pi(a)e, re\rangle = rf(a),\ a = 1 \Rightarrow t = r,$$

so that $g_1 = f$. Similarly, $g_2 = f$ and f is a pure state. □

Existence of faithful representations

We now want to show that the positive linear functionals separate points of A in order to obtain a faithful representation via an orthogonal direct sum of cyclic representations associated to positive functionals discussed above.

Proposition 3 ([A], Theorem 1.7.2)**.** *For every self-adjoint $a \in A$, there is a state f such that $|f(a)| = \|a\|$. Moreover, f can be taken pure.*

Proof. Let $f \in C^*(a)'$ be a homomorphism at which $|\Gamma(a)|$ takes its maximum $\|a\|$, so that $|f(a)| = \|a\|$. Any such f satisfies $f(1) = 1 = \|f\|$. By Hahn-Banach, f extends to a linear functional on A with $\|f\| = 1$. By Lemma 1, f is a state.

Let F be the collection of states g such that$g(a) = f(a) = \|a\|$ (f and a as above). Then F is a closed, convex subset of the weak* unit ball, hence compact and equal to the closed convex hull of its extreme points (Banach-Alaoglu and Krein-Milman). Let g be an extreme point of F. We want to show that g is also an extreme state (pure). If $g = tg_1 + (1-t)g_2$, with $g_1, g_2 \in S_A$ and $0 < t < 1$, then $g_1, g_2 \in F$ as follows. Since $\|g_i\| = 1$, we have $|g_i(a)| \le \|a\| = |g(a)|$ and

$$\|a\| = |f(a)| = |g(a)| = |tg_1(a) + (1-t)g_2(a)| \le \|a\|,$$

so that $g_i(a) = g(a)$ and $g_i \in F$. Hence $g = g_i$ since g is extreme in F. □

Applying the above to a^*a, we get a (pure) state f such that $f(a^*a) = \|a^*a\| = \|a\|^2$. If π_f is the cyclic (irreducible) representation with unit generator e obtained from the GNS construction, then

$$\|\pi_f(a)e\|_f^2 = \langle \pi_f(a)e, \pi_f(a)e\rangle_f = \langle \pi_f(a^*a)e, e\rangle_f = f(a^*a) = \|a\|^2,$$

and $\|\pi_f(a)\| = \|a\|$.

For each $a \in A$, let f_a be a state such that $\|\pi_{f_a}(a)\| = \|a\|$ as just discussed. Taking an orthogonal direct sum, we obtain a faithful representation

$$\bigoplus_{a\in A} \pi_{f_a} : A \to \mathcal{B}\left(\bigoplus_{a\in A} \mathcal{H}_{f_a}\right),$$

achieving our goal. If we take pure states f_a, then the right hand side is an orthogonal direct sum of irreducible representations. Finally, if A is separable we can let a range over a countable dense subset of A, so that $\bigoplus_{a\in A} \mathcal{H}_{f_a}$ is a separable Hilbert space.

Appendix: Loose ends

For completeness we include the following equivalences, some of which were used above.

Proposition 4 ([D], Proposition 2.3.1)**.** *The following are equivalent:*

1. *The representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is irreducible,*
2. *Every $0 \ne x \in \mathcal{H}$ is cyclic for π ($\{\pi(a)x : a \in A\}$ is dense in $\mathcal{H}$),*
3. *The commutant of $\pi(A)$ is $\mathbb{C}$.*

Proof. The first and second statements are equivalent since any element of a proper non-trivial closed invariant subspace won't be cyclic, and the orbit closure of a a non-zero non-cyclic vector will define a proper non-trivial closed subspace.

The first and third statements are equivalent since a proper-non-trivial closed subspace defines a projection $\ne 0, 1$ that commutes with $\pi(A)$, and if the commutant is larger than $\mathbb{C}$ it will contain projections that define closed invariant subspaces (e.g. spectral projections of self-adjoint elements of the commutant, [C] Theorem IX.2.2). □

Proposition 5 ([D], Proposition 1.6.1 or [C], Theorem VIII.3.6 or [A] Proposition 1.8.1)**.** *The following are equivalent, and define the* ***positive*** *elements A_+ of A:*

1. $a = b^*b$,

2. $a = b^2$ *for some* $b = b^*$,

3. $a = a^*$ *and* $\sigma(a) \subseteq [0, \infty)$.

Moreover, the positive elements form a closed cone in A*, i.e.* A_+ *is closed and*

$$\{\alpha p + \beta q : p, q \in A_+,\ 0 \leq \alpha, \beta \in \mathbb{R}\} \subseteq A_+,$$

with $A_+ \cap -A_+ = \{0\}$.

Proof. The second and third statements are equivalent by the functional calculus; if $\sigma(a) \subseteq [0, \infty)$, we can take a positive square root (which will be self-adjoint), and if $a = b^2$ with $b = b^*$, then $\sigma(a) = \sigma(b)^2 \geq 0$.

Clearly the second statement implies the first, so to finish we show that the first statement implies the second. First note that there is a decomposition $b^*b = u^2 - v^2$ with $uv = vu = 0$ and u, v self-adjoint, where $u = f(b^*b)$, $v = g(b^*b)$ and

$$f(t) = \left\{ \begin{array}{cc} \sqrt{t} & t \geq 0 \\ 0 & t \leq 0 \end{array} \right. ,\ g(t) = \left\{ \begin{array}{cc} 0 & t \geq 0 \\ \sqrt{-t} & t \leq 0 \end{array} \right.$$

(sometimes called the orthogonal decomposition or Hahn decomposition). We want to show that $v = 0$. With $w = bv$ we have

$$w^*w = v^*b^*bv = v^*(u^2 - v^2)v = -v^4,$$

so that $\sigma(w^*w) =^(v^2)^2 \leq 0$. Now let $w = s + it$ with s, t self-adjoint (in general, any $a \in A$ satisfies $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$). Then

$$ww^* = (s + it)(s - it) + (s - it)(s + it) - w^*w = 2s^2 + 2t^2 + v^4 \in A_+$$

since A_+ is a cone (still to be shown). So $\sigma(ww^*) \geq 0$ and $\sigma(w^*w) \leq 0$. [We now note that for any $a, b \in A$, the non-zero spectrum of the products, $\sigma(ab) \setminus \{0\}$ and $\sigma(ba) \setminus \{0\}$, are equal. This is proved as follows. Suppose $\lambda \neq 0$ and $(\lambda - ab)^{-1} = c$. Then

$$\begin{aligned} (\lambda - ba)(1 + bca) &= \lambda - ba + b(\lambda - ab)ca = \lambda, \\ (1 + bca)(\lambda - ba) &= \lambda - ba + bc(\lambda - ab)a = \lambda, \end{aligned}$$

and $(\lambda - ba)^{-1} = \frac{1}{\lambda}(1 + bca)$.] Hence $\sigma(w^*w) = \sigma(ww^*) = \sigma(v^4) = \{0\}$ and we have $v = 0$ (any self-adjoint element with zero spectrum must be zero by the functional calculus).

Finally, we need to show that A_+ is a cone (obviously closed under any of the definitions above), for which it suffices to show that if $a, b \in A_+$ then $a + b \in A_+$, positive scaling obviously preserving A_+. To this end, assume $\|a\|, \|b\| \leq 1$. Then $\sigma(1 - a), \sigma(1 - b) \in [0, 1]$, $\|1 - a\|, \|1 - b\| \leq 1$, $\|1 - \frac{a+b}{2}\| \leq 1$, and $\sigma(a + b) \subseteq [0, 2]$. [Here we've used the facts that the spectral radius doesn't exceed the norm and that the spectral radius of a self-adjoint element is the norm of the element.] □

References

[A] William Arveson, *An Invitation to C^*-Algebras*, Graduate Texts in Mathematics, Vol. 39, Springer.

[C] John B. Conway, *A Course in Functional Analysis*, Second Edition, Graduate Texts in Mathematics, Vol. 96, Springer.

[D] Jacques Dixmier, *Les C^*-algèbres et leurs Représentations*, Gauthier-Villars, 1969.

[GN] I. M. Gelfand, M. A. Naimark, *On the imbedding of normed rings into the ring of operators on a Hilbert space*, Matematicheskii Sbornik 12 (2) (1943), 197–217.

[S] I. E. Segal, *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc. 53 (1) (1947), 73–88.