Integrating trigonometric functions

Here are the most basic trigonometric identities

$$\sin^2 x + \cos^2 x = 1 \text{ (Pythagorean identity)}$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

From these follow

$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x.$$

and

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \ \sin^2 x = \frac{1-\cos(2x)}{2}.$$

We can use these along with our other integration techniques to integrate new functions. [Appendix C of the text contains a review of trigonometry.] You should also memorize the following:

$$\int \sec x \ dx = \ln|\sec x + \tan x|, \ \int \csc x \ dx = -\ln|\csc x + \cot x|.$$

1. Integrate $\int \sin x \cos x \ dx$ two ways, with a substitution and with a trig identity.

   Here are three ways: Let $u = \sin x$, $du = \cos x \ dx$ to get

   $$\int \sin x \cos x \ dx = \int u \ du = u^2/2 + C = \sin^2 x/2 + C.$$

   Let $u = \cos x$, $du = -\sin x \ dx$ to get

   $$\int \sin x \cos x \ dx = -\int u \ du = -u^2/2 + C = -\cos^2/2 + C.$$

   Use $\sin(2x) = 2\sin x \cos x$ to get

   $$\int \sin x \cos x \ dx = \frac{1}{2}\int \sin(2x) dx = -\cos(2x)/4 + C.$$

   Exercise: Use trig identities to show that these answers are all equivalent (i.e. differ by a constant).

2. $\int \cos^5 x \sin^4 x \ dx$

   Let $u = \sin x$, $du = \cos x \ dx$ and use $\cos^2 x - 1 = \sin^2 x$ to get

$$\int \cos^5 x \sin^4 x \ dx = \int (\cos^2)^2 x \sin^4 x \cos x \ dx = \int (1-\sin^2 x)^2 \sin^4 x \cos x \ dx$$
$$= \int (1-u^2)^2 u^4 du = \int (u^4 - 2u^6 + u^8) du = \sin^5 x/5 - 2\sin^7 x/7 + \sin^9 x/9 + C.$$

3. $\displaystyle\int \sin^3 x\ dx$

We have (with $u = \cos x$, $du = -\sin x\ dx$)

$$\int \sin^3 x\ dx = \int (1-\cos^2 x)\sin x\ dx = -\int (1-u^2)du = u-u^3/3+C = \cos x - \cos^3 x/3+C.$$

4. $\displaystyle\int \sec^4 x \tan^3 x\ dx$

With $u = \sec x$, $du = \sec x \tan x\ dx$, we have

$$\int \sec^4 x \tan^3 x\ dx = \int \sec^3 x \tan^2 x \sec x \tan x\ dx = \int \sec^3 x(\sec^2 x - 1)\sec x \tan x\ dx$$
$$= \int u^3(u^2-1)du = u^6/6 - u^4/4 + C = \sec^6 x/6 - \sec^4 x/4 + C.$$

5. $\displaystyle\int \sec^2 x \tan^3 x\ dx$

With $u = \tan x$, $du = \sec^2 x\ dx$, we get

$$\int \sec^2 x \tan^3 x\ dx = \int u^3\ du = u^4/4 + C = \tan^4 x/4 + C.$$

6. $\displaystyle\int \cot^3 x \csc^3 x\ dx$

With $u = \csc x$, $du = -\csc x \cot x\ dx$, we get

$$\int \cot^3 x \csc^3 x\ dx = \int \cot^2 x \csc^2 x \csc x \cot x\ dx = \int (\csc^2 x - 1)\csc^2 x \csc x \cot x\ dx$$
$$= -\int (u^2-1)u^2 du = -u^5/5 + u^3/3 + C = -\csc^5 x/5 + \csc^3 x/3 + C.$$

7. $\displaystyle\int \sec x\ dx,\ \int \csc x\ dx$

$$\int \sec x\ dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.$$

With $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)dx$ we have

$$\int \sec x\ dx = \int \frac{du}{u} = \ln|u| = \ln|\sec x + \tan x|.$$

Use a similar trick for $\int \csc x\ dx$.

8. $\int \cos^4 x \, dx$

$$\int \cos^4 x \, dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos(2x) + \cos^2(2x))dx$$
$$= \frac{x}{4} + \sin(2x) + \int \frac{1+\cos(4x)}{2} dx = \frac{x}{4} + \sin(2x) + \frac{x}{2} + \frac{\sin(4x)}{8}.$$

9. $\int \cos^2 x \sin^2 x \, dx$

There are a few ways to do this using different trigonometric identities, but here is one way.

$$\int \cos^2 x \sin^2 x \, dx = \frac{1}{4}\int \sin^2(2x)dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2} dx = \frac{x}{8} - \frac{\sin(4x)}{32}.$$

10. $\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1)dx = \tan x - x.$$