Taylor polynomial remainder

Suppose f is $(n+1)$-times differentiable on the interval between a and a fixed number x. We want to show that

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt \\ &= T_n(x) + R_n(x), \end{aligned}$$

where $T_n(x)$ is the nth Taylor polynomial for f (centered at a) and the nth remainder $R_n(x)$ is given by the integral

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

This explicitly tells us the difference between a function and its Taylor polynomial. Here is a proof.

The fundamental theorem of calculus tells us that

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Integrating by parts with

$$u = f'(t),\ du = f''(t)dt,\ dv = dt,\ v = t - x,$$

gives

$$\begin{aligned} f(x) - f(a) &= f'(t)(t-x)\Big|_a^x + \int_a^x (x-t)f''(t)dt = -f'(a)(x-a) + \int_a^x f''(t)(x-t)dt, \\ f(x) &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)dt = T_1(x) + R_1(x). \end{aligned}$$

Integrating the remainder R_1 by parts, with

$$u = f''(t),\ du = f'''(t)dt,\ dv = (x-t)dt,\ v = -\frac{(x-t)^2}{2},$$

gives

$$\begin{aligned} R_1(x) &= \int_a^x f''(t)(x-t)dt = -f''(t)\frac{(x-t)^2}{2}\Big|_a^x + \frac{1}{2}\int_a^x f'''(t)(x-t)^2 dt \\ &= \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x f'''(t)(x-t)^2 dt. \end{aligned}$$

So we get

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + R_1(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x f'''(t)(x-t)^2 dt \\ &= T_2(x) + R_2(x). \end{aligned}$$

Repeatedly integrating by parts n times will give the desired result:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt \\ &= T_n(x) + R_n(x). \end{aligned}$$

Taylor's inequality

Now we want to estimate the size of the remainder, R_n. Suppose that for t in the interval between the center a and the fixed number x we know

$$|f^{(n+1)}(t)| \leq M,$$

i.e. we can bound the $(n+1)$st derivative on the interval between a and x. Then

$$\begin{aligned} |R_n(x)| &= \left| \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt \right| \leq \frac{M}{n!} \left| \int_a^x (x-t)^n dt \right| \\ &= \frac{M}{n!} \left| \frac{(x-t)^{n+1}}{n+1} \Big|_a^x \right| = \frac{M}{n!} \frac{|x-a|^{n+1}}{n+1} \\ &= \frac{M}{(n+1)!} |x-a|^{n+1}. \end{aligned}$$

This inequality,

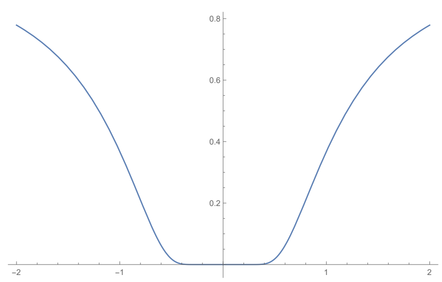
$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1},$$

known as Taylor's inequality, will be useful to us in the sequel.

A non-analytic C^∞ function

Here is an example of an infinitely differentiable function that cannot be expressed as a power series near zero:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$



As you can see, this function is very "flat" at $x = 0$; in fact we have $f^{(n)}(0) = 0$, every derivative of f at zero is equal to zero. So the Taylor series for f centered at $a = 0$ is $\sum_{n=0}^{\infty} 0 \cdot x^n = 0$, the zero function. In particular, f is not equal to its Taylor series, so f cannot be written as a power series centered at zero.

Problems

- Give a bound for the remainder $R_n(x)$ between $\sin x$ and $T_n(x)$ centered at $a = 0$ (this will depend on n and x). What happens as $n \to \infty$? Do the same for $\cos x$ centered at $a = 0$.

- What is the $(n+1)$st derivative of e^x? How big can this be on the interval between $a = 0$ and x? (Treat the cases $x \geq 0$ and $x \leq 0$ separately.) Find the appropriate bound for the remainder. What happens as $n \to \infty$?

- Use the first, second, and third degree Taylor polynomial for $\sqrt{1+x}$ to estimate $\sqrt{1.1}$. How good is your estimate in each case?