MATH 2300-004 QUIZ 9 Name: ____________________________

1. Given a function $f(x)$ that is infinitely differentiable at $x = a$, what is its Taylor series centered at a?

   The Taylor series is
   $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$
   If f can be expressed as a power series near $x = a$, then that power series must be the Taylor series.

2. [Memorization] What are the Taylor series for the following functions (centered at zero)?

   (a) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

   (b) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

   (c) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (d) $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

   (e) $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

3. For this problem, let $f(x) = (1+x)^{1/3}$

(a) Find $f'(x)$, $f''(x)$, and $f'''(x)$.

We have

$$f'(x) = \left(\frac{1}{3}\right)(1+x)^{-2/3}$$
$$f''(x) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(1+x)^{-5/3}$$
$$f'''(x) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(1+x)^{-8/3}.$$

(b) What is the maximum M of $|f'''(x)|$ on the interval $[0,1]$?

The function $|f'''(x)| = \frac{10}{27(1+x)^{7/3}}$ is decreasing on the interval $[0,1]$ hence attains its maximum at $x = 0$, $|f'''(0)| = \frac{10}{27} = M$.

(c) What is $T_2(x)$, the second degree Taylor polynomial for f centered at $x = 0$?

The second degree Taylor polynomial is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{x}{3} - \frac{x^2}{9}.$$

(d) Use $T_2(x)$ to estimate $\sqrt[3]{2}$.

$\sqrt[3]{2} = f(1)$ so we estimate using $T_2(1) = 1 + 1/3 - 1/9 = 11/9$.

(e) Bound the absolute value of the remainder $R_2(1) = f(1) - T_2(1) = \sqrt[3]{2} - T_2(1)$ using Taylor's inequality and the bound M on $|f'''(x)|$ you found above.

Taylor's inequality states that

$$|R_2(1)| \leq \frac{M}{3!}|1-0|^3 = \frac{5}{81}$$

so that

$$\frac{94}{81} \leq \sqrt[3]{2} \leq \frac{104}{81}.$$