MATH 2300-004 QUIZ 8 Name: ____________________

Collaborators (if any):

Due Monday, April 1st at the beginning of class. Submit your work on additional paper, treating this page as a cover sheet. You may use technology and work with with other students. If you work with others, please list their names above.

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1. Find the interval of convergence for each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$

Solution. The ratio test gives the condition

$$|x| \lim_{n\to\infty} \frac{1/(2(n+1))!}{1/(2n)!} = |x| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1,$$

which is always true, so that $R = \infty$ and the power series converges on the whole real line, $(-\infty, \infty)$.

[Remark: This power series agrees with $\cos(\sqrt{x})$ for $x \geq 0$ and gives a natural way to make sense of $\cos(\sqrt{x})$ for negative values of x.]

(b) $\displaystyle\sum_{n=0}^{\infty} n^n (x+1)^n$

Solution. The ratio of the absolute value of consecutive terms is

$$|x+1| \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} = |x+1| \lim_{n\to\infty} (n+1)(1+1/n)^n = \begin{cases} 0 & x = -1 \\ \infty & x \neq -1 \end{cases}$$

so that $R = 0$ and the power series converges only at $x = -1$ where it is centered.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (x-1)^n$

Solution. The radius of convergence R is determined by

$$\begin{aligned} |x-1| \lim_{n\to\infty} \frac{(n+1)2^{n+1}/\sqrt{(n+1)^3+1}}{n2^n/\sqrt{n^3+1}} \\ = |x-1| \lim_{n\to\infty} 2\frac{n+1}{n}\sqrt{\frac{n^3+1}{(n+1)^3+1}} \\ = 2|x-1| < 1,\ |x-1| < \frac{1}{2}, \end{aligned}$$

so that $R = 1/2$. We must check convergence at the endpoints $x = 1/2, 3/2$. At $x = 1/2$ the power series evaluates to

$$\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (1/2-1)^n = \sum_{n=0}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

which diverges by limit comparison to $\sum_n \frac{1}{\sqrt{n}}$. At $x = 3/2$ the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$$

which converges by the alternating series test (note that the derivative of $x/\sqrt{x^3+1}$ is $\frac{2-x^3}{2(x^3+1)^{3/2}}$ which is negative for $x > \sqrt[3]{2}$, so the absolute value of the terms of the series are decreasing to 0). Hence the interval of convergence is $(1/2, 3/2]$.

2. Starting with the geometric series, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$, show that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

[Hint: Evaluate $\arctan(x)$ at $1/\sqrt{3}$.]

Solution. We have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|x| < 1$. Integrating term-by-term gives

$$\arctan x = \int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x| < 1$, after noting that the two functions agree at $x = 0$. Evaluating at $x = 1/\sqrt{3}$, we get

$$\pi/6 = \arctan(1/\sqrt{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)},$$

which gives the desired result after multiplying both sides by 6.

3. For this problem, let $f(x) = (1+x)^{1/3}$

   (a) Find $f'(x)$, $f''(x)$, and $f'''(x)$.
   We have

$$\begin{aligned} f'(x) &= \left(\frac{1}{3}\right)(1+x)^{-2/3} \\ f''(x) &= \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(1+x)^{-5/3} \\ f'''(x) &= \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(1+x)^{-8/3}. \end{aligned}$$

   (b) What is $T_3(x)$, the third degree Taylor polynomial for f centered at $a = 0$?

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$$

(c) Use $T_3(x)$ to estimate $\sqrt[3]{2}$.

$$\sqrt[3]{2} = \sqrt[3]{1+1} \approx T_3(1) = 1 + 1/3 - 1/9 + 5/81 = 104/81.$$

4. For this problem, you may assume $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (i.e. e^x is equal to its Taylor series centered at zero).

(a) Integrate term-by-term to evaluate the definite integral

$$\int_0^1 e^{-x^2}\,dx.$$

[Your answer will be an infinite series.]
Solution. We have

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\Big|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = 1 - 1/3 + 1/10 - 1/42 + \dots$$

(b) Use the alternating series remainder estimate to to give an approximation to the above integral so that the absolute value of the error is less than 0.001.
Solution. The series obtained above is a convergent alternating series. The remainder after adding N terms is bounded by the $(N+1)$st term, i.e.

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} - \sum_{n=0}^{N} \frac{(-1)^n}{n!(2n+1)}\right| \leq \frac{1}{(N+1)!(2N+3)}.$$

We want the remainder to be less than 0.001, so we find the smallest such N satisfying

$$\frac{1}{(N+1)!(2N+3)} < 0.001,$$

which is $N = 4$. In other words,

$$\frac{5651}{7560} - 0.001 \leq \int_0^1 e^{-x2}dx \leq \frac{5651}{7560} + 0.001$$

(where $5651/7560 = 1 - 1/3 + 1/10 - 1/42 + 1/216$ is the sum up to $N = 4$).

5. Find the Taylor series for $\cos x$ centered at $a = \pi/2$ in two ways:

(a) from the definition (i.e. calculate $\frac{d^k}{dx^k}\Big|_{x=\pi/2} \cos x$),
Solution. We need the derivatives for $\cos x$,

$$\cos x,\ -\sin x,\ -\cos x,\ \sin x \text{ (four-periodic)},$$

evaluated at $\pi/2$,

$$0,\ -1,\ 0,\ 1 \text{ (four-periodic)}.$$

The Taylor series is then

$$-(x-\pi/2)+\frac{1}{3!}(x-\pi/2)^3-\frac{1}{5!}(x-\pi/2)^5+\ldots,$$

or in summation notation

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}.$$

(b) using the identity $\cos x=\cos(x-\pi/2+\pi/2)$ and the Taylor series for $\sin x$ centered at $a=0$. [You may assume that $\sin x$ is equal to its Taylor series.]

Solution. We have

$$\cos x=\cos(x-\pi/2+\pi/2)=\cos(x-\pi/2)\cos(\pi/2)-\sin(x-\pi/2)\sin(\pi/2)=-\sin(x-\pi/2).$$

Hence we can take the Taylor series for $\sin x$ centered at zero, multiply by -1, and substitute $x-\pi/2$ for x to obtain the Taylor series for $\cos x$ centered at $\pi/2$:

$$\cos x=-\sin(x-\pi/2)=-\sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}.$$