MATH 2300-004 QUIZ 6 Name: ____________________

1. What is a sequence? What does it mean for a sequence to converge? Give an example of a convergent sequence and of a divergent sequence.

A sequence $(a_n)_n$ is an ordered list of real numbers. A sequence converges,

$$\lim_{n\to\infty} a_n = L,$$

if there is a real number L to which the a_n are approaching as n increases, i.e. a_n becomes arbitrarily close to L for n sufficiently large.

[More precisely, the sequence a_n converges to L if for any error $\epsilon > 0$ (no matter how small!), there is an index $N > 0$ (possibly very big!) such that $|a_n - L| < \epsilon$ when $n \geq N$. In other words, when you go out further than N terms in the sequence, the terms of the sequence are within the specificed small error around L.]

Here are some convergent sequences and their limits:

$$(a_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right),$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$(b_n)_{n=0}^{\infty} = \left(\cos\left(\frac{\pi}{2^n}\right)\right)_{n=0}^{\infty} = \left(-1, 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}, \ldots\right),$$
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\left(\frac{\pi}{2^n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{2^n}\right) = \cos(0) = 1,$$
$$(c_n)_{n=1}^{\infty} = \left(1 + \frac{(-1)^n}{n}\right)_{n=1}^{\infty} = \left(0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots\right),$$
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1.$$

Here are some divergent sequences:

$$(a_n)_{n=0}^{\infty} = (2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \ldots)$$

exponential growth,

$$(b_n)_{n=0}^{\infty} = \left(\sin\left(\frac{n\pi}{3}\right)\right)_{n=0}^{\infty} = \left(0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \ldots\right)$$

periodic sequence,

$$(c_n)_{n=0}^{\infty} = \left((-1)^n(n^2+1)\right)_{n=0}^{\infty} = (1, -2, 5, -10, 17, \ldots)$$

grows in absolute value, alternating in sign.

2. What is a series? What does it mean for a series to converge? Give an example of a convergent series and of a divergent series.

A series $\sum_{n=0}^{\infty} a_n$ is (a formal expression for) an attempt to add infinitely many numbers a_n, the terms of the series. A series is convergent if the limit of the partial sums

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} (a_0 + a_1 + \ldots + a_{N-1} + a_N)$$

exists. In other words, we add up finitely many terms (in order) from a given sequence a_n and see if the partial sums approach a limiting value.

Here are some convergent series:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = \frac{1}{1 - 1/2} = 2,$$

a convergent geometric series, $r = 1/2$, $|r| < 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots = 1,$$

a telescoping series, or convergent by comparison to $\sum_{n} \frac{1}{n^2}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6},$$

a convergent p-series, $p = 2 > 1$. [The fact that the sum is $\pi^2/6$ is a non-trivial fact we won't show in class.]

Here are some divergent series:

$$\sum_{n=0}^{\infty} 2^n, \text{ a divergent geometric series, } r = 2, \ |r| \geq 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \text{ a divergent } p\text{-series}, p = 1/2 \leq 1,$$

$$\sum_{n=0}^{\infty} \frac{2n^2 - 1}{3n^3 + 1}, \text{ divergent by comparison to the harmonic series } \sum_{n} \frac{1}{n}.$$

3. Use the integral test to determine the convergence or divergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

The function $f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing on the interval $[2, \infty)$ (positive because x and $(\ln x)^2$ are positive there, decreasing because the numerator is fixed and the denominator is increasing). The series therefore behaves like the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{T\to\infty} \int_2^T \frac{dx}{x(\ln x)^2} = \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{du}{u^2}$$
$$= \lim_{T\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln T} = \lim_{T\to\infty} -\frac{1}{\ln T} + \frac{1}{\ln 2} = \frac{1}{\ln 2} < \infty.$$

The improper integral converges, therefore the series converges as well.

4. Use the comparison test to determine the convergence or divergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n - n^3}{5^n + n^5}$.

We can use either the direct comparison test or the limit comparison test, in both cases comparing to the convergent geometric series $\displaystyle\sum_{n=0}^{\infty} (3/5)^n$.

For the direct comparison test, we have

$$0 \le \frac{3^n - n^3}{5^n + n^5} \le \left(\frac{3}{5}\right)^n,$$

so the series in the problem statement is dominated by the convergent geometric series $\sum_n (3/5)^n$.

For the limit comparison test, we have

$$\lim_{n\to\infty} \frac{\frac{3^n - n^3}{5^n + n^5}}{(3/5)^n} = \lim_{n\to\infty} \frac{1 - \frac{n^3}{3^n}}{1 + \frac{n^5}{5^n}} = 1$$

since

$$\lim_{n\to\infty} \frac{n^5}{5^n} = \lim_{n\to\infty} \frac{n^3}{3^n} = 0$$

e.g. using L'Hôpital's rule a few times. Therefore the series in the problem statement has the same convergence behavior as the convergent geometric series $\sum_n (3/5)^n$.