MATH 2300-004 QUIZ 5 Name: ____________________________

Collaborators (if any):

Due Friday, February 22nd at the beginning of class. Submit your work on additional paper, treating this page as a cover sheet. You may use technology and work with with other students. If you work with others, please list their names above. SHOW YOUR WORK!

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1. Do exercise 21, section 6.6 of the text.

   We are lifting disks of water to the top of the tank. The weight of each disk is $62.5\pi r^2 dx$ lbs (weight density times volume). Measuring distance x ft from the top of the tank gives

   $$\text{Work} = \int \text{Force} \times \text{Distance} = \int_0^8 (62.5\pi r^2 dx)(x).$$

   We can write the radius of a disk x ft from the top in terms of x using similar triangles (draw a picture)

   $$\frac{3}{8} = \frac{r-3}{8-x}, \quad r = 6 - \frac{3}{8}x.$$

   Hence

   $$W = 62.5\pi \int_0^8 x(6 - 3x/8)^2 dx = 33000\pi = 103672.557 \text{ ft-lbs.}$$

2. Find the centroid of the region bounded by the given curves.

   (a) $y = \cos x,\ y = \sin x,\ \pi/4 \le x \le 3\pi/4.$

   The total area is:

   $$M = \int_{\pi/4}^{3\pi/4} (\sin x - \cos x) dx = -\cos x - \sin x\Big|_{\pi/4}^{3\pi/4} = \sqrt{2}.$$

   The moment about the y-axis is (with density $\rho = 1$)

   $$M_y = \int_{\pi/4}^{3\pi/4} x(\sin x - \cos x) dx = \sin x - x\cos x - x\sin x - \cos x\Big|_{\pi/4}^{3\pi/4} = \frac{4+\pi}{2\sqrt{2}},$$

   (using integration by parts). The moment about the x-axis is:

   $$M_x = \int_{\pi/4}^{3\pi/4} \frac{\sin^2 x - \cos^2 x}{2} dx = -\frac{\sin(2x)}{4}\Big|_{\pi/4}^{3\pi/4} = \frac{1}{2},$$

   (integrating with the identity $\cos^2 x - \sin^2 x = \cos(2x)$ or otherwise). Hence the centroid is

   $$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(1 + \frac{\pi}{4}, \frac{1}{2\sqrt{2}}\right).$$

   (b) $y = 1/x^3,\ y = 0,\ 1 \le x < \infty.$

   The total area is:

   $$M = \int_1^\infty \frac{1}{x^3} dx = -\frac{1}{2x^2}\Big|_1^\infty = \frac{1}{2}.$$

The moment about the y-axis is (with density $\rho = 1$)

$$M_y = \int_1^\infty x \frac{1}{x^3}\, dx = -\frac{1}{x}\bigg|_1^\infty = 1.$$

The moment about the x-axis is:

$$M_x = \int_1^\infty \frac{1}{2}\left(\frac{1}{x^3}\right)^2 dx = -\frac{1}{10}x^{-5}\bigg|_1^\infty = \frac{1}{10}.$$

Hence the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(2, \frac{1}{5}\right).$$

3. Determine whether the sequence converges or diverges. If it converges, find its limit.

(a) $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

Intuition: The denominator grows exponentially like $(e^2)^n$ while the numerator grows exponentially like e^n. Hence the quotient decays exponentially like $(1/e)^n$ and should go to zero as $n \to \infty$. To make this more rigorous, divide the numerator and denominator by e^{2n} to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{-n} + e^{-3n}}{1 - e^{-2n}} = \frac{0+0}{1+0} = 0.$$

(b) $b_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

Combing the logarithms and dividing the numerator and denominator of the argument by n^2 gives

$$b_n = \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\frac{2 + 1/n^2}{1 + 1/n^2}\right).$$

Hence

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{2+1/n^2}{1+1/n^2}\right) = \ln\left(\lim_{n\to\infty}\frac{2+1/n^2}{1+1/n^2}\right) = \ln\left(\frac{2+0}{1+0}\right) = \ln 2.$$

(c) $c_n = \sqrt[n]{2^n + 3^n}$

Intuition: 2^n is inconsequential when compared to 3^n, so the sequence behaves like $(3^n)^{1/n} = 3$. To make this rigorous, we factor out 3^n and take logarithms

$$\ln(c_n) = \frac{1}{n}\ln(3^n(1 + (2/3)^n)) = \ln 3 + \frac{1}{n}\ln(1 + (2/3)^n) \to \ln 3 + 0 = \ln 3.$$

Hence $\lim_{n\to\infty} c_n = e^{\ln 3} = 3$.

Another approach is to "squeeze" the terms of the sequence,

$$3 = (3^n)^{1/n} \le (2^n + 3^n)^{1/n} \le (3^n + 3^n)^{1/n} = 2^{1/n}3.$$

As $n \to \infty$, $2^{1/n} \to 1$ (take logarithms if you're not convinced). Hence

$$3 = \lim_{n\to\infty} 3 \le \lim_{n\to\infty}(2^n + 3^n)^{1/n} \le 3 \lim_{n\to\infty} 2^{1/n} = 3.$$

(d) $d_n = \dfrac{\sin(n)\ln n}{n}$

Intuition: $\sin n$ is bounded by 1 and $\ln n$ grows much more slowly than n, so the sequence should converge to zero. To make this rigorous, we note that

$$|d_n| \le \frac{\ln n}{n}$$

and that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \text{``}\frac{\infty}{\infty}\text{''} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

using L'Hôpital's rule. Hence

$$0 = -\lim_{n\to\infty} \frac{\ln n}{n} \le \lim_{n\to\infty} d_n \le \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

and the limit is zero as expected.

(e) $e_n = \left(1 + \dfrac{t}{n}\right)^n$, where t is a constant.

This sequence converges to e^t (and you should know this, perhaps taking it for a definition of e^t). Taking logarithms gives

$$\lim_{n\to\infty} \ln e_n = \lim_{n\to\infty} n\ln(1+t/n) = \lim_{n\to\infty} \frac{\ln(1+t/n)}{1/n} = \text{``}\frac{0}{0}\text{''},$$

an indeterminant form. We use L'Hôpital's rule

$$\lim_{x\to\infty} \frac{\ln(1+t/x)}{1/x} = \lim_{x\to\infty} \frac{(-t/x^2)/(1+t/x)}{-1/x^2} = \lim_{x\to\infty} \frac{t}{1+t/x} = t,$$

to conclude that

$$\lim_{n\to\infty} e_n = e^t.$$

4. Show the following:

(a) For any $\epsilon > 0$, $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = 0$. I.e., $\ln x$ grows more slowly than any power of x.

The limit is indeterminant, ∞/∞. Applying L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = \lim_{x\to\infty} \frac{1/x}{\epsilon x^{\epsilon-1}} = \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} = 0.$$

(b) For any $p > 0$, $\displaystyle\lim_{x\to\infty} \frac{x^p}{e^x} = 0$. I.e., e^x (or a^x for any $a > 1$) grows more quickly than any power of x.

Let n be the integer such that $n - 1 < p \le n$. Applying L'Hôpital's rule n times gives

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{p(p-1)\cdot\ldots\cdot(p-n+1)x^{p-n}}{e^x} = \lim_{x\to\infty} \frac{p(p-1)\cdot\ldots\cdot(p-n+1)}{x^{n-p}e^x} = 0$$

since $n - p \geq 0$.
Another approach is to take *pth* roots first:

$$\lim_{x\to\infty}\left(\frac{x^p}{e^x}\right)^{1/p} = \lim_{x\to\infty}\frac{x}{e^{x/p}} \overset{L'H}{=} \lim_{x\to\infty}\frac{1}{e^{x/p}/p} = 0.$$

So

$$\lim_{x\to\infty}\frac{x^p}{e^x} = 0^p = 0,\ \ p > 0.$$