[Quiz 4, Due Monday, February 11th] Name:

1. Integrate:

   (a) $\displaystyle\int \frac{y^2}{(1-y^2)^{3/2}}dy$

   With $y = \sin\theta$, $dy = \cos\theta\ d\theta$ the integral becomes

$$\int \frac{\sin^2\theta\cos\theta\ d\theta}{(1-\sin^2\theta)^{3/2}} = \int \tan^2\theta\ d\theta = \int (\sec^2\theta - 1)d\theta = \theta - \tan\theta$$
$$= \arcsin y - \tan(\arcsin y) = \arcsin y - \frac{y}{\sqrt{1-y^2}}.$$

   (b) $\displaystyle\int_0^{\pi/3} \sec^3\theta\tan\theta\ d\theta$

   With $u = \sec\theta$, $du = \sec\theta\tan\theta\ d\theta$ the integral becomes

$$\int_1^2 u^2\ du = u^3/3\Big|_1^2 = 7/3.$$

   (c) $\displaystyle\int \frac{x^2+8x+18}{(x+3)^3}dx$

   First find the partial fraction decomposition of the integrand:

$$\frac{x^2+8x+18}{(x+3)^3} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3},$$

$$x^2+8x+18 = A(x+3)^2 + B(x+3) + C = Ax^2 + (6A+B)x + (9A+3B+C),$$

$$\text{(compare coefficients and solve) } A = 1,\ B = 2,\ C = 3.$$

   Next, integrate:

$$\int \left(\frac{1}{x+3} + \frac{2}{(x+3)^2} + \frac{3}{(x+3)^3}\right) dx = \ln|x+3| - \frac{2}{x+3} - \frac{3/2}{(x+3)^2}.$$

   One could also set $u = x+3$, $x = u-3$, simplify, and integrate.

   (d) $\displaystyle\int \ln(\sin t)\sin t\cos t\ dt$

   With $x = \sin t$, $dx = \cos t\ dt$, the integral becomes $\int x\ln x\ dx$ which we integrate by parts,

$$u = \ln x,\ dv = x\ dx,\ du = dx/x,\ v = x^2/2,$$

$$\int x\ln x\ dx = \frac{x^2\ln x}{2} - \int \frac{x}{2}dx = \frac{x^2\ln x}{2} - \frac{x^2}{4}$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) = \frac{\sin^2 t}{2}\left(\ln(\sin t) - \frac{1}{2}\right).$$

2. Let R be the unbounded region in the fourth quadrant between the curves

$$x = 0,\ y = 0,\ y = \ln x.$$

Find the volume of the solid obtained by rotating R around

(a) the x-axis,

(b) the y-axis.

Note that these are both improper integrals!

For part (a), cross-sections perpendicular to the x-axis being disks of radius $y = |\ln x|$, the volume is

$$\int_0^1 \pi(\ln x)^2 dx.$$

The indefinite integral to compute is (using parts once or twice)

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\ dx = x(\ln x)^2 - 2(x\ln x - x).$$

For the improper integral (improper since $\ln x$ is unbounded near $x = 0$) we take a limit

$$\int_0^1 \pi(\ln x)^2 dx = \pi \lim_{\epsilon\to 0^+} (x(\ln x)^2 - 2(x\ln x - x))\Big|_\epsilon^1 = \pi \lim_{\epsilon\to 0^+} (2 - \epsilon(\ln\epsilon)^2 + 2\epsilon\ln\epsilon - 2\epsilon).$$

We can find the limits of the two "interesting" summands with L'Hôpital's rule

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{1/x} \overset{L'H}{=} -2\lim_{x\to 0^+} x\ln x = -2\lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{L'H}{=} 2\lim_{x\to 0^+} x = 0.$$

Hence the volume is 2π.

For part (b), cross-sections perpendicular to the y-axis being disks of radius $x = e^y$, the volume is

$$\int_{-\infty}^0 \pi(e^y)^2 dy.$$

For the improper integral we take a limit

$$\begin{aligned}\int_{-\infty}^0 \pi(e^y)^2 dy &= \pi \lim_{T\to -\infty} \int_T^0 e^{2y} dy = \pi \lim_{T\to -\infty} \frac{e^{2y}}{2}\Big|_T^0 \\ &= \pi \lim_{T\to -\infty} \frac{1}{2} - \frac{e^{2T}}{2} = \frac{\pi}{2},\end{aligned}$$

since $e^{2T} \to 0$ as $T \to -\infty$.