[Quiz 3, Friday February 1st] Name:

$$\text{Find } \int \frac{dx}{x^3-1} \text{ (use partial fractions).}$$

The denominator of the integrand factors as $x^3 - 1 = (x-1)(x^2+x+1)$. We want to solve

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Clear denominators, expand,

$$1 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C),$$

and equate coefficients to get the system of linear equations

$$\begin{aligned} A + B &= 0 \\ A - B + C &= 0 \\ A - C &= 1. \end{aligned}$$

The solution is $A = 1/3$, $B = -1/3$, and $C = -2/3$.

So we are integrating

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx.$$

The first integral is straightforward,

$$\int \frac{dx}{x-1} = \ln|x-1|.$$

For the second integral, complete the square in denominator, $x^2+x+1 = (x+1/2)^2 + 3/4$, and make a substitution $u = x + 1/2$:

$$\int \frac{x+2}{x^2+x+1}dx = \int \frac{x+2}{(x+1/2)^2+3/4}dx = \int \frac{u+3/2}{u^2+(\sqrt{3}/2)^2}du.$$

This gives (with $v = u^2 + 3/4$)

$$\int \frac{u+3/2}{u^2+(\sqrt{3}/2)^2}du = \frac{1}{2}\int \frac{dv}{v} + \frac{3}{2}\int \frac{du}{u^2+(\sqrt{3}/2)^2}.$$

The first integral is again straightforward

$$\int \frac{dv}{v} = \ln|v| = \ln|u^2+3/4| = \ln|x^2+x+1|.$$

The second integral requires knowing

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a}\arctan(t/a),$$

so that

$$\int \frac{du}{u^2+(\sqrt{3}/2)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}/2}\right) = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$

Putting everything together gives

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$